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# Critical fluctuations of QCD phase transitions and their related observables

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**Abstract.** We study how the dilepton production rate and the electric conductivity are affected by the soft modes inherently associated with the second-order phase transitions at the critical temperature of color superconductivity and QCD critical point. It is shown that the soft modes modify the photon self-energy significantly through the so called Aslamasov-Larkin, Maki-Thompson and density of states terms, which are known responsible for the paraconductivity in the metallic superconductivity, and they lead to an anomalous enhancement of the production rate in the low energy/momemtum region and the divergence of conductivity at the critical temperature.

## 1 Introduction

Rich phase structure is expected to exist in the high baryon-density region of the Quantum Chromodynamics (QCD), which will be revealed by various experimental programs in relativistic heavy-ion collisions (HIC) such as the beam energy scan program at RHIC, HADES and NA61/SHINE, as well as the future experiments at FAIR, NICA and J-PARC-HI. In this proceeding, we show a possible observability of the color superconducting (CSC) [1] phase transition and QCD critical point (CP) in these experiments through an anomalous enhancement of the dilepton production rate (DPR) caused by the soft modes associated with these phase transitions [2, 3]. The low energy limit of the DPR is related to the electric conductivity. We calculate the effects of the soft modes on the DPR and electric conductivity by extending the theory of the paraconductivity in metals [4]. It is shown that the soft modes lead to an anomalous enhancement of the DPR at low energy region and the electric conductivity near each phase transition. We argue that these enhancements are used for experimental observables to verify the existence of the second-order phase transition on the QCD phase diagram.

## 2 Formalism

To study the temperature and quark chemical potential region around the 2-flavor color superconductivity (2SC) and QCD CP, we employ the 2-flavor NJL model

$$\mathcal{L} = \bar{\psi}i(\gamma^{\mu}\partial_{\mu} - m)\psi + G_{\mathrm{S}}[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\vec{\tau}\psi)^2] + G_{\mathrm{C}}(\bar{\psi}i\gamma_5\tau_2\lambda_A\psi^C)(\bar{\psi}^C i\gamma_5\tau_2\lambda_A\psi), \quad (1)$$

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where $\vec{\tau} = (\tau_1, \tau_2, \tau_3)$ is the Pauli matrix for the flavor $SU(2)_f$, $\lambda_{A=2,5,7}$ is the antisymmetric component of Gell-Mann matrix for color $SU(3)_c$ and $\psi^C(x) \equiv C\bar{\psi}^T(x)$ with $C = i\gamma_2\gamma_0$. We consider the chiral limit for the analysis of the DPR near the critical temperature of the 2SC, while we set the curren quark mass to $m = 4$ MeV for investigating the QCD CP. The scalar coupling constant $G_S = 5.01$ MeV$^{-2}$ and the three-momentum cutoff $\Lambda = 650$ MeV are used, while the diquark coupling $G_C$ is treated as a free parameter.

The DPR is related to the retarded photon self-energy $\Pi^{R\mu\nu}(k)$ as

$$\frac{d^4\Gamma}{d^4k} = -\frac{\alpha}{12\pi^4}\frac{1}{k^2}\frac{1}{e^{\omega/T}-1}g_{\mu\nu}\mathrm{Im}\Pi^{R\mu\nu}(k), \tag{2}$$

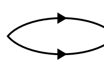
where $k = (\mathbf{k}, \omega)$ is the four momentum of the photon and $\alpha$ is the fine structure constant. The electric conductivity is given in terms of the low energy limit of $\mathrm{Im}\Pi^{R\mu\nu}(k)$ by

$$\sigma = \frac{1}{3}\lim_{\omega\to 0}\frac{1}{\omega}\sum_{i=1,2,3}\mathrm{Im}\Pi^{Rii}(\mathbf{0}, \omega). \tag{3}$$

In this section, we first calculate the photon self-energy $\tilde{\Pi}^{\mu\nu}(k)$ including the diquark soft modes assosiated with the 2SC phase transition [2] in Sec. 2.1, and then deal with the soft modes of the QCD CP in Sec. 2.2. The DPR and electric conductivity calculated from this photon self-energy through Eqs. (2) and (3) reflect the effect of the soft mode.

## 2.1 Modification of $\Pi^{\mu\nu}$ by diquark soft modes

It is known that the diquark fluctuations form a collective mode with a significant strength near but above $T_c$ of the 2SC [6, 7]. The collective mode associated with the second-order phase transition is called the soft mode. The propagator $\tilde{\Xi}_C(q)$ of the diquark soft modes in the random-phase approximation (RPA) in the imaginary-time formalism is given by

$$\tilde{\Xi}_C(k) = \frac{1}{{G_C}^{-1} + \mathcal{Q}_C(k)}, \quad \mathcal{Q}_C(k) =$$  $$= -8\int_p \mathrm{Tr}[\mathcal{G}_0(k-p)\mathcal{G}_0(p)], \tag{4}$$

where $\mathcal{Q}_C(k) = \mathcal{Q}_C(\mathbf{k}, i\nu_n)$ is the one-loop $qq$ correlation function with the free quark propagator $\mathcal{G}_0(p) = \mathcal{G}_0(\mathbf{p}, i\omega_m) = 1/[(i\omega_m + \mu)\gamma_0 - \mathbf{p}\cdot\gamma]$, the Matsubara frequency for fermions (bosons) $\omega_m$ ($\nu_n$), and the trace over the Dirac indices Tr. The retarded propagator is obtained by the analytic continuation $\Xi_C^R(\mathbf{k}, \omega) = \tilde{\Xi}_C(\mathbf{k}, i\nu_n \to \omega + i\eta)$.

We remark that $\Xi_C^R(\mathbf{k}, \omega)$ satisfies the Thouless criterion, that is $[\Xi_C^R(\mathbf{0}, 0)]^{-1} = 0$ at $T = T_c$ for the second-order phase transition. The criterion shows that the pole of $\Xi_C^R(\mathbf{k}, \omega)$ continuously approaches the origin of the complex energy plane as $T$ goes toward $T_c$, and hence the diquark fluctuations become soft near $T_c$ [6]. This fact also allows us to adopt the time-dependent Ginzburg-Landau (TDGL) approximation

$$[\Xi_C^R(\mathbf{k}, \omega)]^{-1} = A(\mathbf{k}) + C(\mathbf{k})\omega, \tag{5}$$

near but above $T_c$, where $A(\mathbf{k}) = G_C^{-1} + Q_C^R(\mathbf{k}, 0)$ and $C(\mathbf{k}) = \partial Q_C^R(\mathbf{k}, \omega)/\partial\omega\,|_{\omega=0}$. It is known that Eq. (5) reproduces $\Xi_C^R(\mathbf{k}, \omega)$ over wide ranges of $\omega$, $\mathbf{k}^2$ and $T(> T_c)$ [2, 6].

To construct $\Pi^{\mu\nu}$ that inolves the soft mode, we start from the one-loop diagram of $\tilde{\Xi}_C(\mathbf{k}, i\nu_n)$, which is the lowest contribution of the soft modes to the thermodynamic potential [2]. The photon self-energy is then constructed by attaching electromagnetic vertices at two points of quark lines in the thermodynamic potential. One then obtains four types of diagrams shown in Fig. 1. These diagrams are called the Aslamasov-Larkin (AL) (Fig. 1(a)),

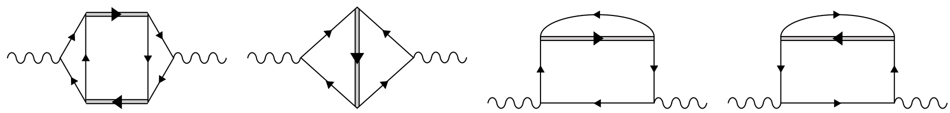



Figure 1: Diagrammatic representations of the Aslamasov-Larkin (a), Maki-Thompson (b) and density of states (c,d) terms with the diquark soft mode. The wavy and double lines are photons and the diquark soft modes, respectively.

Maki-Thompson (MT) (Fig. 1(b)) and density of states (DOS) (Fig. 1(c, d)) [4] terms, respectively, in the theory of metallic superconductivity. Each contribution to the photon self-energy, $\tilde{\Pi}^{\mu\nu}_{\rm AL}(k)$, $\tilde{\Pi}^{\mu\nu}_{\rm MT}(k)$ and $\tilde{\Pi}^{\mu\nu}_{\rm DOS}(k)$, respectively, is given by

$$\tilde{\Pi}^{\mu\nu}_{\rm AL}(k) = 3\int_q \tilde{\Gamma}^{\mu}(q,q+k)\tilde{\Xi}_C(q+k)\tilde{\Gamma}^{\nu}(q+k,q)\tilde{\Xi}_C(q), \tag{6}$$

$$\tilde{\Pi}^{\mu\nu}_{\rm MT(DOS)}(k) = 3\int_q \tilde{\Xi}_C(q)\mathcal{R}^{\mu\nu}_{\rm MT(DOS)}(q,k), \tag{7}$$

where $\tilde{\Gamma}^{\mu}(q,q+k)$ and $\mathcal{R}^{\mu\nu}(q,k) = \mathcal{R}^{\mu\nu}_{\rm MT}(q,k)+\mathcal{R}^{\mu\nu}_{\rm DOS}(q,k)$ satisfy the following Ward-Takahashi (WT) identities

$$k_\mu\tilde{\Gamma}^{\mu}(q,q+k) = (e_u+e_d)[Q_C(q+k)-Q_C(q)], \tag{8}$$

$$k_\mu\mathcal{R}^{\mu\nu}(q,k) = (e_u+e_d)[\tilde{\Gamma}^{\nu}(q-k,q)-\tilde{\Gamma}^{\nu}(q,q+k)], \tag{9}$$

where $e_u = 2|e|/3$ ($e_d = -|e|/3$) is the electric charge of up (down) quarks. The total photon self-energy is then given by

$$\tilde{\Pi}^{\mu\nu}(k) = \tilde{\Pi}^{\mu\nu}_{\rm free}(k) + \tilde{\Pi}^{\mu\nu}_{\rm fluc}(k), \tag{10}$$

$$\tilde{\Pi}^{\mu\nu}_{\rm fluc}(k) = \tilde{\Pi}^{\mu\nu}_{\rm AL}(k) + \tilde{\Pi}^{\mu\nu}_{\rm MT}(k) + \tilde{\Pi}^{\mu\nu}_{\rm DOS}(k), \tag{11}$$

where $\tilde{\Pi}^{\mu\nu}_{\rm free}(k)$ is the contribution of the free quark gases and $\tilde{\Pi}^{\mu\nu}_{\rm fluc}(k)$ is the one of the soft modes. One can explicitly check that this photon self-energy satisfies the WT identity using Eqs. (8) and (9).

To obtain the DPR with Eq. (2), the calculation of the spatial components of $\Pi^{R\mu\nu}(k)$ is sufficient since $\Pi^{R00}(k)$ is expressed in terms of the longitudinal part from the WT identity as $\Pi^{R00}(k) = \mathbf{k}^2\Pi^{R11}(k)/k_0^2$ with $k = (k_0, |\mathbf{k}|, 0, 0)$. For the vertex functions (8) and (9), we approximate $\tilde{\Gamma}^{\mu}(q,q+k)$ and $\mathcal{R}^{\mu\nu}(q,k)$ using Eq. (5). For $\tilde{\Gamma}^{i}(q,q+k)$, we employ an ansatz

$$\tilde{\Gamma}^{i}(q,q+k) = -(e_u+e_d)\frac{Q_C(\mathbf{q}+\mathbf{k},0)-Q_C(\mathbf{q},0)}{|\mathbf{q}+\mathbf{k}|^2-|\mathbf{q}|^2}(2q+k)^i, \tag{12}$$

which is a real number and satisifes the WT identity (8) with Eq. (5). One can also obtain $\mathcal{R}^{ij}(q,k)$ in the same manner, and finds that it is a real number as well. Using this vertex and Eq. (5), the imaginary part of $\Pi^{Rij}_{\rm MT}(q) + \Pi^{Rij}_{\rm DOS}(q)$ vanishes, which is in accordance with the case of the metallic superconductivity [4]. Therefore, the MT and DOS terms do not contribute to the calculation of the DPR, and hence we can calculate the contribution of the soft mode on the DPR through only the AL term. We note that $\tilde{\Pi}^{\mu\nu}_{\rm fluc}(k)$ obtained by using the approximations (5) and (12) is satisfied with the WT identity of the photon self-energy.

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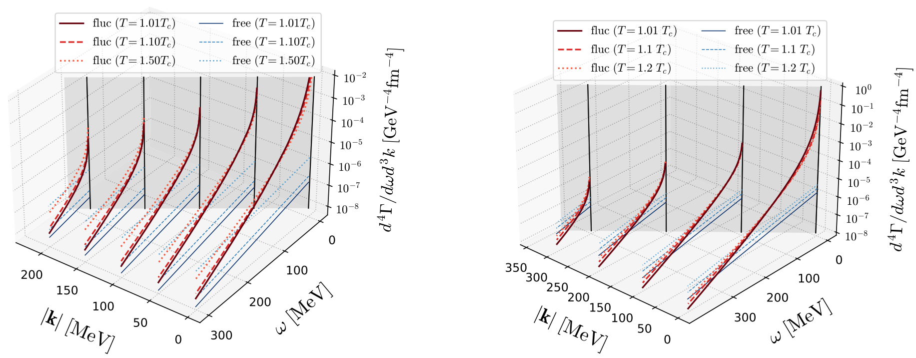


Figure 2: DPRs per unit energy $\omega$ and momentum $\mathbf{k}$ above $T_c$ of the 2SC at $\mu = 350$ MeV (left) [2] and of the QCD CP at $\mu = \mu_{\rm CP}$ (right) [3]. The thick lines are the contribution of the soft modes, and the thin lines are the results for the free quark gas.

### 2.2 Modification of $\Pi^{\mu\nu}$ by soft modes near QCD CP

The photon self-energy including the soft modes of the QCD CP can also be calculated with a similar manner [3]. The propagator of this mode is evaluated by the RPA as

$$\tilde{\Xi}_S(k) = \frac{1}{{G_S}^{-1} + Q_S(k)}, \quad Q_S(k) = \text{(loop diagram)} = 12\int_p \mathrm{Tr}[\mathcal{G}_0(p-k)\mathcal{G}_0(p)], \tag{13}$$

where $Q_S(k)$ is the one-loop $\bar{q}q$ correlation function. As described in Ref. [5], $\Xi_S^R(k)$ has a discontinuity at the light cone, and its strength is limited to the space-like region. This property is contrasted with the diquark soft mode (4), which does not have the discontinuity at the light cone, while diquark soft mode also has a significant strength in the space-like region. The photon self-energy with this soft modes is given by

$$\tilde{\Pi}^{\mu\nu}(k) = \tilde{\Pi}^{\mu\nu}_{\rm AL}(k) + \tilde{\Pi}^{\mu\nu}_{\rm MT}(k) + \tilde{\Pi}^{\mu\nu}_{\rm DOS}(k),$$
$$\tilde{\Pi}^{\mu\nu}_{\rm AL}(k) = \sum_f \int_q \tilde{\Gamma}^{\mu}_f(q, q+k)\tilde{\Xi}_S(q+k)\tilde{\Gamma}^{\nu}_f(q+k, q)\tilde{\Xi}_S(q), \tag{14}$$
$$\tilde{\Pi}^{\mu\nu}_{\rm MT(DOS)}(k) = \sum_f \int_q \tilde{\Xi}_S(q)\mathcal{R}^{\mu\nu}_{f\ {\rm MT(DOS)}}(q, k), \tag{15}$$

where $f = u, d$ is the index of the flavor. The vertices $\tilde{\Gamma}^{\mu}_f(q, q+k)$ and $\mathcal{R}^{\mu\nu}_f(q,k) = \mathcal{R}^{\mu\nu}_{f\ {\rm MT}}(q,k) + \mathcal{R}^{\mu\nu}_{f\ {\rm DOS}}(q,k)$ satisfy the WT identity, similarly to Eqs. (8) and (9).

## 3 Numerical results and summary

The left panel of Fig. 2 shows the DPR $d^4\Gamma/d^4k$ per unit energy $\omega$ and momentum $\mathbf{k}$ near the 2SC phase transition for $T = 1.01T_c$, $1.1T_c$ and $1.5T_c$ at $\mu = 350$ MeV ($T_c = 42.94$ MeV) [2]. The thick lines are the contribution from the soft modes, while the thin lines are the ones of the free quark gas. One sees that the DPR from the soft modes is anomalously enhanced at the relatively small $\omega$ and $\mathbf{k}$ region in comparison with the free quark gas for $T \lesssim 1.5T_c$, and this enhancement is more pronounced as $T$ approaches $T_c$. This result is expected through the properties of the soft modes. Shown in the right panel of Fig. 2 is the DPR near the QCD CP

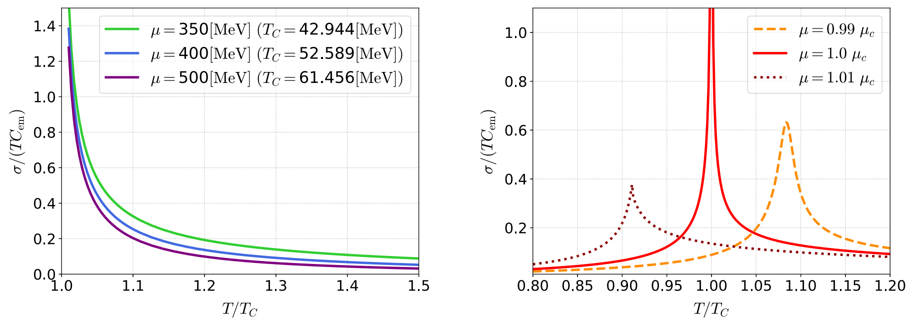


Figure 3: The electric conductivity $\sigma$ assosiated with the soft modes. **Left**:$T$ dependence of $\sigma$ assosiated with the diquark soft modes for $\mu$ = 350 MeV, 400 MeV and 500 MeV [2]. **Right**: $T$ dependence of $\sigma$ assosiated with the soft modes of the QCD CP for $\mu/\mu_{\rm CP}$ = 0.99, 1.0 and 1.01 [3].

for $T = 1.01T_{\rm CP}$, $1.1T_{\rm CP}$ and $1.2T_{\rm CP}$ at $\mu = \mu_{\rm CP}$, where $(T_{\rm CP}, \mu_{\rm CP}) = (50.156, 322.12)$ MeV is the location of the QCD CP. One finds that the anomalous enhancement of the DPR is also observed similarly to the case of the 2SC [3].

The magnitude of the DPR around the origin is related to the electric conductivity $\sigma$ through Eqs. (2) and (3). We show the $T$ dependence of $\sigma$ in Fig. 3. The left (right) panel is $\sigma$ of the 2SC (the QCD CP), where $\sigma$ is normalized by $T$ and $C_{\rm em} = (e_u + e_d)^2$ in the left panel ($C_{\rm em} = e_u^2 + e_d^2$ in the right panel). One can see that the conductivities are divergent at the critical temperature in both system. We find that these quantities diverge at $T_c$ with $\sigma \propto \epsilon^{-1/2}$ for the 2SC, while for the case of the QCD CP, $\sigma \propto \epsilon^{-2/3}$ with $\epsilon = |T - T_c|/T_c$, as will be discussed in the forthcoming publication. The origin of the difference of two cases is due to the $T$-dependence of $A(\mathbf{k})$ and the $\mathbf{k}$-dependence of $C(\mathbf{k})$ in Eq. (5).

In this report, we studied how the soft modes of the 2SC and QCD CP affect the DPR and the electric conductivity around the phase transitions. The modification of the photon self-energy due to the soft modes are investigated through the analysis of the Aslamasov-Larkin, Maki-Thompson and density of states terms with the TDGL approximation for the soft mode propagator and vertices, which is satisfied with the WT identity of the photon self-energy and vertices. The enhancement of the DPR found in this study would allow us to detect these signals of the 2SC and QCD CP in the future HIC experiments.

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