Phases and condensates in zero-temperature QCD at finite \( \mu_I \) and \( \mu_S \)

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Abstract. I discuss pion and kaon condensation and the properties of the phases of QCD at finite isospin chemical potential \( \mu_I \) and strangeness chemical potential \( \mu_S \) at zero temperature using three-flavor chiral perturbation theory. Electromagnetic effects are included in the calculation of the phase diagram, which implies that the charged meson condensed phases become superconducting phases of QCD with a massive photon via the Higgs mechanism. Without electromagnetic effects, we show results for the light quark condensate and the pion condensate as functions of \( \mu_I \) at next-to-leading (NLO) order in the low-energy expansion. The results are compared with recent lattice simulations and by including the NLO corrections, one obtains very good agreement.

1 Introduction

In this talk, I would like to discuss various aspects of the phases of QCD at zero temperature, but finite isospin and strangeness density. However, before I do that, I will briefly comment on the QCD phase diagram as it is normally presented, namely in the \( \mu_B-T \) plane. It is shown in Fig. 1, borrowed from Ref. [1]. Few of the results for the phase diagram are rigorous in the sense that they are obtained from first principles, rather they are obtained by model calculations. However, for asymptotically high temperatures and zero baryon chemical potential, we know that QCD is in a quark-gluon plasma phase consisting of weakly interacting deconfined quarks and gluons. Similarly, we know that at asymptotically high baryon density and zero temperature, QCD is in the color-flavor locked phase arising from an attracting channel of one-gluon exchange and the resulting instability of the Fermi surface. From lattice simulations, we know that there is a cross-over transition for \( \mu_B = 0 \) at a temperature of around 155 MeV. For low temperatures and large chemical potentials, the infamous sign problem, prohibits the use of standard Monte Carlo techniques to study the properties of QCD. One must therefore resort to low-energy models such as the NJL model and the quark-meson model. Over the past two decades, a huge amount of work has been done to map out the phase diagram.

The situation is even more complex than this since, instead of using a common quark chemical potential for all quarks, one can introduce a separate chemical potential \( \mu_f \) for each
flavor. For three flavors, we use either $\mu_u$, $\mu_d$, and $\mu_s$ or the baryon chemical potential $\mu_B$, the isospin chemical potential $\mu_I$, and strangeness chemical potential $\mu_S$ defined as

$$\mu_B = \frac{3}{2}(\mu_u + \mu_d), \quad \mu_I = \mu_u - \mu_d, \quad \mu_S = \frac{1}{2}(\mu_u + \mu_d - 2\mu_s).$$

For $\mu_B = \mu_S = 0$ but nonzero $\mu_I$, one can carry out Monte Carlo simulations using standard techniques since the fermion determinant in this case is real and consequently there is no sign problem. This opens up the possibility to study charged pion condensation on the lattice and confront it with results from low-energy effective theories. In this talk, I will discuss pion condensation for two and three flavors using chiral perturbation theory ($\chi$PT) as a low-energy effective theory for QCD and show results for the light quark and pion condensates as a function of $\mu_I$ with $\mu_B = \mu_S = 0$. The results will be compared to recent high-precision lattice simulations [2–5]. I will also discuss the phase diagram and meson condensation for three flavors in the $\mu_I-\mu_S$ plane at zero temperature, with and without electromagnetic interactions.

![Phase diagram of QCD in the $\mu_B-T$ plane. Figure from Ref. [1].](image)

In Fig. 2, we sketch the phase diagram of two-flavor QCD in the the $\mu_I-T$ plane. In the lower left region, we have the hadronic phase where chiral symmetry is broken and quarks are confined. As the temperature increases, one enters the quark-gluon plasma phase. Along the $\mu_I$-axis, there is a transition from the hadronic phase to a Bose-condensed phase of charged pions. In this phase, the $U(1)_I$ symmetry is broken giving rise to a massless Goldstone boson, which is a mixture of $\pi^+$ and $\pi^-$. For large isospin chemical and low temperature, one expects that quarks are the relevant degrees of freedom rather than pions [6]. The Fermi surface that exists when the interactions are turned off, is rendered unstable once they are turned on, since they are attractive. The system is then described in terms of loosely bound Cooper pairs instead of tightly bound pions. Since the symmetry breaking pattern is the same, there is a cross-over transition rather than a true phase transition between the BEC and the BCS phases [6].

2 $\chi$PT at finite isospin chemical potential $\mu_I$

We will be using chiral perturbation theory to describe the pion-condensed phase of QCD at finite $\mu_I$ and zero temperature. $\chi$PT is a low-energy effective theory for QCD based on
the (global) symmetries and degrees of freedom. It provides a correct model-independent description as long as one is inside its domain of validity [7–9]. The effective Lagrangian that describes the low-energy degrees of freedom of QCD (pions, kaons, eta) can be written in a low-energy expansion,

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_4 + \ldots,$$

(2)

where the subscript denotes the order in the expansion. The expansion parameter can be written as \( \frac{M}{f} \) with \( M \) being a relevant mass or momentum scale and \( f \) is the pion-decay constant. The leading-order Lagrangian is the nonlinear sigma model, which for two flavors reads

$$\mathcal{L}_2 = \frac{1}{4} f^2 \langle \nabla_\mu \Sigma \nabla^\mu \Sigma^+ \rangle + \frac{1}{4} f^2 \langle \chi^+ \chi + \Sigma^+ \Sigma \rangle,$$

(3)

where \( \langle \rangle \) means trace in flavor space and

$$\Sigma = e^{i \phi_a / f}, \quad \chi = 2B_0 \text{diag}(m_u, m_d),$$

(4)

with \( \phi_a \) being the Goldstone bosons fields, \( m_{u,d} \) are the quark masses, and \( B_0 \) is the related to the tree level quark condensate in the vacuum via \( \langle \bar{\psi} \psi \rangle = -f^2 B_0 \). The covariant derivative is

$$\nabla_\mu \Sigma = \partial_\mu \Sigma - i \left[ v_\mu, \Sigma \right],$$

(5)

with \( v_\mu = \delta_\mu 0(\frac{1}{4} \mu B \frac{1}{2} + \frac{1}{4} \mu T_3) \). Note that our results will be independent of \( \mu_B \) since the unit matrix \( 1 \) commutes with everything, this reflects that the mesons have zero baryon number.

The most general ansatz for the normalized ground state can after some simplifications be written as [6]

$$\Sigma_\alpha = 1 \cos \alpha + i \tau_2 \sin \alpha = e^{i \alpha \tau_2} = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix},$$

(6)

which simply is a rotation in flavor space of the vacuum state \( 1 \) by an angle \( \alpha \). The leading-order thermodynamic potential is

$$\Omega_0 = -f^2 B_0 (m_u + m_d) \cos \alpha - \frac{1}{2} f^2 \mu_I^2 \sin^2 \alpha.$$

(7)
We see that there is a competition between the two terms in Eq. (7), the first term prefers the vacuum state $\alpha = 0$, while the second term prefers $\alpha = \frac{1}{2} \pi$. The optimum is found by balancing these two terms in the thermodynamic potential,

$$
\cos \alpha = \frac{m_{\pi,0}^2}{\mu_I^2}, \quad \mu_I^2 \geq m_{\pi,0}^2, \quad (8)
$$

$$
\alpha = 0, \quad \mu_I^2 < m_{\pi,0}^2, \quad (9)
$$

where $m_{\pi,0}^2 = B_0(m_\pi + m_\rho)$ is the tree-level pion mass. Thus there is a phase transition from the vacuum to a pion-condensed phase at a critical $\mu_I$, $\mu_I^2 = \pm m_{\pi,0}$. To determine the order of the transition, one can construct a Ginzburg-Landau energy functional by expanding the thermodynamic potential $\Omega_0$ around $\alpha = 0$,

$$
\Omega_0 = -f^2 m_{\pi,0}^2 + \frac{1}{2} f^2 [m_{\pi,0}^2 - \mu_I^2 \alpha^2 - \frac{1}{24} f^2 (m_{\pi,0}^2 - 4\mu_I^2)] \alpha^4 + O(\alpha^6). \quad (10)
$$

We define a critical chemical isospin potential $\mu_I^c$ when the order-$\alpha^2$ term vanishes, i.e. $\mu_I^c = \pm m_{\pi,0}$. Since the coefficient of the order-$\alpha^4$ term is positive for $\mu_I = \mu_I^c$, the transition is second order. Note that all thermodynamic quantities are independent of $\mu_I$ for $\mu_I^2 < m_{\pi,0}^2$ implying that e.g. the isospin density vanishes in the same region and not only for $\mu_I = 0$. This is an example of the Silver-Blaze property [10].

### 3 Phase diagram for three flavors

We next consider the phase diagram for three-flavor QCD at finite $\mu_I$ and $\mu_S$ including electromagnetic effects. If we couple $\chi$PT to dynamical photons, the Lagrangian contains a few extra terms at leading order in the low-energy expansion [11],

$$
\mathcal{L}_2 = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{4} f^2 (\nabla_\mu \Sigma \nabla^\mu \Sigma^\dagger) + \frac{1}{4} f^2 (\chi^\dagger \Sigma + \Sigma^\dagger \chi) + C \langle Q \Sigma Q \Sigma^\dagger \rangle + \mathcal{L}_{\text{gf}} + \mathcal{L}_{\text{ghost}}, \quad (11)
$$

with $\chi = 2 B_0 \text{diag}(m_\mu, m_\tau, m_\pi)$ and $\Sigma = e^{i \theta_{\mu\tau}}$. The new term $C \langle Q \Sigma Q \Sigma^\dagger \rangle$ is responsible for the tree-level mass splitting of the charged and neutral pions, and it also contributes to the tree-level mass splitting between the charged and neutral kaons. The inclusion of electromagnetic effects also implies that the phases with charged meson condensates are superconducting and that the massless degree of freedom (the Goldstone boson) is eaten up by the photon which becomes massive via the Higgs mechanism. The term $v_\mu$ in the covariant derivative Eq. (5) is replaced by

$$
v_0 = \frac{1}{3} (\mu_B - \mu_S) a_3 + \frac{1}{2} \mu_{K^0} \lambda_Q + \frac{1}{2} \mu_{K^+} \lambda_K, \quad v_i = 0, \quad (12)
$$

where

$$
\mu_{K^+} = \frac{1}{2} \mu_I + \mu_S, \quad \mu_{K^0} = -\frac{1}{2} \mu_I + \mu_S, \quad (13)
$$

$$
\lambda_Q = \lambda_3 + \frac{1}{\sqrt{3}} \lambda_8, \quad \lambda_K = -\lambda_3 + \frac{1}{\sqrt{3}} \lambda_8. \quad (14)
$$

In analogy with the two-flavor case, we expect onset of charged kaon condensation when $\mu_{K^+}^2 = m_{K^+}^2$ and neutral kaon condensation when $\mu_{K^0}^2 = m_{K^0}^2$. The corresponding ansätze for
the ground states are \( 1 \Sigma_\beta = e^{i\beta \lambda_5} = \begin{pmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{pmatrix}, \quad \Sigma_\gamma = e^{i\gamma \lambda_7} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \gamma & \sin \gamma \\ 0 & -\sin \gamma & \cos \gamma \end{pmatrix}. \tag{15} \)

The thermodynamic potential in the different phases can then be computed as functions of the chemical potentials. For example, in the charged kaon condensed phase, the thermodynamic potential is

\[
\Omega_0 = -f^2 B_0 (m_u + m_s) \cos \alpha - \frac{1}{2} f^2 \left[ \mu_{K^+}^2 - \Delta m_{EM}^2 \right] \sin^2 \alpha, \tag{16}
\]

where \( \Delta m_{EM}^2 = \frac{2e^2}{f^2} \) is the splitting between the charged and neutral kaons due to electromagnetism. It follows that the transition takes place exactly at \( \mu_{K^+}^2 = m_{K^+}^2 = B_0 (m_u + m_d) + \frac{2e^2}{f^2} \) as expected.

For each value of \((\mu_I, \mu_S)\), we find the phase with the lowest value of \( \Omega_0 \) (or largest pressure). This phase wins and we can map out the phase diagram in this manner. The result is shown in the left panel of Fig. 3. The black lines are the transition lines without electromagnetic interactions and the red lines are with electromagnetic interactions. The former was first obtained by Kogut and Toublan [12] in the isospin limit. The phases with charged meson condensation become Higgs phases upon including electromagnetic effects, with a tree-level mass of the photon of \( m_A = ef \sin \alpha \). The transitions from the normal phase to a meson-condensed phase is always second order with mean-field exponents in the \( O(2) \) universality class. The transitions between the various condensed phases are always first order and involve the competition between the order parameters of the different phases. As we cross the transition lines, the order parameters as well as the isospin and strangeness densities, \( n_I \) and \( n_S \) jump discontinuously. The small offset of the dashed vertical lines is due to the mass difference between the charged and neutral kaons, which is both due to \( \Delta m_{EM} \neq 0 \), and \( m_u \neq m_d \). These contributions, however, pull in opposite directions, as we see in the phase diagram. The contribution due to the difference in quark masses adds to the mass of the \( K^0/\bar{K}^0 \) meson, which is why the black transition line between the k\( \text{aon condensate is to the left of the } \mu_I = 0 \text{ line, while the electromagnetic contribution adds to the mass of the charged kaon, which is why the red line is between these two lines. The partition function in the normal phase is independent of the two chemical potentials } \mu_I \text{ and } \mu_S \text{, which again is the Silver Blaze property [10]. In the right panels, we have zoomed in on the triple points. Upper panel shows the intersection of the normal, neutral kaon condensed and charged kaon condensed phases, while the lower shows the intersection of the normal, pion condensed, and charged kaon condensed phases.}

4 \( O(p^4) \) calculation of thermodynamic potential

I will next sketch the NLO calculation of the thermodynamic potential. For simplicity I consider two flavors in the isospin limit, \( m_u = m_d = m \). The thermodynamic potential can be calculated in a low-energy expansion,

\[
\Omega = \Omega_0 + \Omega_1 + ..., \tag{17}
\]
Figure 3. Left panel shows the phase diagram as predicted by $\chi$PT in the $\mu_I-\mu_S$-plane. In the right panel, we have zoomed in on the triple points. See main text for details. Fig. from Ref. [13].

where $\Omega_n$ is the order-$O(\mu^{2n+2})$ contribution. The term $\Omega_1$ receives contributions from the one-loop graphs of $L_{\text{quadratic}}^2$ and counterterms coming from $L_{\text{static}}^4$. The relevant terms are

$$L_{\text{quadratic}}^2 = \frac{1}{2}(\partial_\mu \phi_a)(\partial^\mu \phi_a) + \mu_I \cos \alpha(\phi_1 \partial_0 \phi_2 - \phi_2 \partial_0 \phi_1)
- \frac{1}{2} \left[(m_{\pi,0}^2 \cos \alpha - \mu_I^2 \cos^2 \alpha)\phi_1^2
+ (m_{\pi,0}^2 \cos \alpha - \mu_I^2 \cos 2\alpha)\phi_2^2
+ (m_{\pi,0}^2 \cos \alpha + \mu_I^2 \sin^2 \alpha)\phi_3^2 \right],
$$

$$L_{\text{static}}^4 = (l_1 + l_2) \mu^4 \sin^4 \alpha + l_4 m_{\pi,0}^2 \mu_I^2 \cos \alpha \sin^2 \alpha + (l_3 + l_4) m_{\pi,0}^4 \cos^2 \alpha + h_1 m_{\pi,0}^4,
$$

where $l_1-l_4$ and $h_1$ are bare couplings. They are related to the renormalized couplings $l'_i$ and $h'_i$ via $l_i = l'_i(\Lambda) + \frac{\gamma_i}{2(4\pi)^2} \left[\frac{1}{\epsilon} + 1\right]$ and $h_i = h'_i(\Lambda) + \frac{\delta_i}{2(4\pi)^2} \left[\frac{1}{\epsilon} + 1\right]$, where $\gamma_i$, $\delta_i$ are coefficients and $\Lambda$ is the renormalization scale in the $\overline{\text{MS}}$ scheme. Since $\delta_1 = 0$, $h_1 = h'_1$ and does not run. Performing the Gaussian integral over the quantum fields $\phi_a$ in dimensional regularization using Eq. (18), we obtain a divergent contribution to $\Omega_1$. The divergences are cancelled by adding the quartic terms from Eq. (19) and renormalizing the couplings by replacing the bare
couplings with the renormalized ones. The final result is

\[
\Omega_0 + \Omega_1 = -f^2 m_{\pi,0}^2 \cos \alpha - \frac{1}{2} f^2 \mu^2 I \sin^2 \alpha
\]

\[
- \frac{1}{4(4\pi)^2} \left[ \frac{3}{2} - \tilde{I}_4 + 4 \tilde{I}_4 + \log \left( \frac{m_{\pi,0}^2}{m_\pi^2} \right) + 2 \log \left( \frac{m_{\pi,0}^2}{m_\pi^2} \right) \right] m_{\pi,0}^4 \cos^2 \alpha
\]

\[
- \frac{1}{4(4\pi)^2} \left[ \frac{1}{2} + \tilde{I}_4 + \log \left( \frac{m_{\pi,0}^2}{m_\pi^2} \right) + \tilde{m}_{\pi,0}^2 f^2 \cos \sin^2 \alpha
\]

\[
- \frac{1}{4(4\pi)^2} \left[ 1 + \frac{2}{3} \tilde{I}_1 + \frac{4}{3} \tilde{I}_2 + 2 \log \left( \frac{m_{\pi,0}^2}{m_\pi^2} \right) \right] \mu_I^4 \sin \pi
\]

\[
- \frac{1}{4(4\pi)^2} \tilde{h}_1 m_{\pi,0}^4 + \tilde{V}_{\text{fin}}^{1,\pi_+} + \tilde{V}_{\text{fin}}^{1,\pi_-},
\]

where \(m_{\pi,0}^2 = m_{\pi,0}^2 \cos \alpha, m_\pi^2 = m_{\pi,0}^2 + \mu^2 \sin^2 \alpha, \) and \(\tilde{V}_{\text{fin}}^{1,\pi_+} + \tilde{V}_{\text{fin}}^{1,\pi_-}\) are two complicated finite terms that must be evaluated numerically. Finally, \(\tilde{I}_4\) and \(\tilde{h}_1\) are, up to a prefactor, equal to \(I_4'\) and \(h_1'\) at the scale \(\Lambda = m_{\pi,0}\). Using Eq. (20) one can show that the phase transition takes place at \(\mu_I^2 = m_{\pi}\), where the physical pion mass \(m_{\pi}\) now includes radiative corrections \([8]\), see Eq. (21) below. The parameters \(\tilde{I}_4\) are determined by experiment and \(\tilde{h}_1\) estimated by model calculations. The parameters \(m_{\pi,0}^2 = 2 B_0 m\) and \(f\) can be found by inverting the one-loop relations using the experimental values for the pion mass and the pion decay constant,

\[
m_{\pi}^2 = m_{\pi,0}^2 \left[ 1 - \frac{m_{\pi,0}^2}{2(4\pi)^2 f^2 I_3} \right], \quad f_{\pi}^2 = f^2 \left[ 1 + \frac{2m_{\pi,0}^2}{(4\pi)^2 f^2 I_1} \right].
\]

### 5 Condensates

In order to obtain the light quark and pion condensates, we need to calculate the thermodynamic potential \(\Omega\) with sources \(m\) and \(j\), where the latter is a pionic source. The former has already been included in the calculations I have shown and it is also straightforward to include a pionic source \(j\) in the calculations. For example, in the two-flavor expression for the thermodynamic potential, one simply makes the replacements \(m \cos \alpha \rightarrow m \cos \alpha + j \sin \alpha\) and \(\bar{h}_1 m_{\pi,0}^4 \rightarrow \bar{h}_1(2 B_0 m)^2 \rightarrow \bar{h}_1[(2 B_0 m)^2 + (2 B_0 j)^2] [14]\. Once these replacements are made, the condensates are given by

\[
\langle \bar{\psi} \psi \rangle_{\mu_I} = \frac{1}{2 \partial \Omega} = -f^2 B_0 \cos \alpha + \ldots, \quad \langle \pi^+ \rangle_{\mu_I} = \frac{1}{2 \partial j} = -f^2 B_0 \sin \alpha + \ldots,
\]

where \(I\) on the the right-hand side have written explicitly the tree-level contributions. The subscript \(\mu_I\) on the expectation values indicates that they depend on the isospin chemical potential. Instead of plotting the condensates directly, we define the normalized deviations as

\[
\Sigma_{\bar{\psi} \psi} = -\frac{2m}{m_{\pi}^2 f_{\pi}^2} \left[ \langle \bar{\psi} \psi \rangle_{\mu_I} - \langle \bar{\psi} \psi \rangle_0 \right] + 1, \quad \Sigma_{\pi} = -\frac{2m}{m_{\pi}^2 f_{\pi}^2} \langle \pi^+ \rangle_{\mu_I}.
\]

At tree level, Eq. (22) shows the rotation of the quark condensate into a pion condensate. Equivalently, from Eq. (23), the deviations at tree level satisfy \(\Sigma_{\bar{\psi} \psi,\text{tree}} + \Sigma_{\pi,\text{tree}} = 1\). This interpretation no longer holds beyond \(O(p^2)\). In the left panel of Fig. 4, we show \(\Sigma_{\bar{\psi} \psi}\) as a function of \(\mu_I/m_{\pi}\) at leading order (red line) \(^2\) and next-to-leading order for two flavors (blue line) and

\(^2\) The leading order result is the same for two and three flavors.
three flavor (green line). The data points are from lattice simulations of Ref. [2–5]. In the right panel, we show $\Sigma$ in the same approximations. We note that the difference between $\Sigma_{\bar{q}q}$ in the various approximations is very small and they all agree very well with the lattice data points. Regarding $\Sigma_\pi$, we notice that it is nonzero for $\mu_I < m_\pi$, which simply reflects that the curves shown are for nonzero pion source, $j = 0.00517054 m_\pi$. The $U(1)_I$-symmetry is therefore broken explicitly for all values of $\mu_I$. Comparing the various approximations and lattice data, it is evident that including the $O(p^4)$ corrections results in a substantially improved agreement between $\chi$PT and simulations. All the numerical results have been obtained by using the same physical meson masses as well as $f_\pi$ as in the lattice simulations. This requires that we invert the relations Eq. (21) to obtain the values for the bare mass $m$ and the bare pion decay constant $f$ using the experimental values of $\bar{l}_i$.

![Figure 4. $\Sigma_{\bar{q}q}$ (left panel) and $\Sigma_\pi$ (right panel) as functions of $\mu_I/m_\pi$ at zero temperature and finite source $j = 0.00517054 m_\pi$. Fig. taken from Ref. [14].](image)

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### References