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# Basic Stellar Physics

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**Abstract.** Polytropes, Virial theorem, evolutionary time scales, degrees of freedom, radiative transport, molecular weight, degeneracy, Jeans mass, and Eddington luminosity are basic ingredients to describe the physics of stars. In the present paper they will be presented in details as long with their role in the stellar evolution.

**PACS.** PACS-key stellar evoluion – PACS-key basic physics

## 1 Introduction

This is not a scientific paper dealing with research results, it is instead thought as a document devoted to students of the European Summer school on Experimental Nuclear Astrophysics who might keep it in their virtual portfolio and consult it, when needed during their present and future studies. Stars evolve by going through successive stages of equilibrium (or non-equilibrium) between the contraction driven by gravity and expansion due to the pressure of gas and radiation fed by thermonuclear burning. Several papers and books describe in details the stellar evolution and nucleosynthesis processes, but the basic ingredients describing those phenomena are few: polytropic relations, Virial theorem, time scales, Jeans mass, degrees of freedom, radiative transport, molecular weight, Eddington limit and degeneracy. In the following paragraphs we will describe how each of these elements of basic physics plays a crucial role in the stellar formation as well as in other stages of the evolution.

## 2 Polytropes

Let us assume for simplicity that stars are made of perfect gases, therefore from the equation of state the gas pressure $P$ can be cexpressed as

$$P = nRT/V \tag{1}$$

where $R = kN_A = 8.3144621 Jmol^{-1}K^{-1}$ is the ideal gas constant, $n$ the number of moles, $k = 1.3806 \cdot 10^{-23} J/K$ the Boltzmann's constant, and $N_A = 6.022 \cdot 10^{23}$ the Avogadro's number. By writing the volume $V$ as a function of the gas density in equation 1 and using the definition of the ideal gas constant,$R$, the pressure $P$ turns out to be:

$$P = \frac{k}{\mu m_H} \rho T \tag{2}$$

where $\mu$ is the molecular weight and $m_H$ is the atomic mass unit.

We can here immediately recognize a particular case of a polytropic transformation [1]. From equation 2 it is clear that the pressure is proportional to the density $\rho$ whenever the temperature $T$ is constant. Therefore, one can write that $P \propto \rho^\delta$, being $\delta = 1$ in case of isothermal processes, where $dT = 0$ and $c$ (the specific heat) is equal to 0. Another special case is the one of adiabatic processes for which $\delta = \gamma$, where $\gamma$ is the adiabatic exponent namely the ratio between the isobaric heat capacity and the isochoric one.

According with Chandrasekhar 1958 [2], a polytrope describes a thermodynamic transformation of a perfect gas, where the energy exchange is proportional to the variation in the temperature: $\delta Q = cdT$, being $T$ the temperature in Kelvin degrees. If the energy is preserved ($\delta Q = 0$) the equation of the adiabatic transformations holds:

$$P \propto A = \pi r^2 V^{-\gamma} \rightarrow P \propto \rho^{\gamma} \tag{3}$$

where the adiabatic exponent $\gamma$ t is equal to 5/3 for a perfect gas with 3 degrees of freedom. $\gamma$ is defined by relation 3, which according with equation 1 can also be written as

$$T \propto \rho^{\gamma-1} \tag{4}$$

By differentiating equations 3 and 4 one obtains

$$\frac{dT}{T} = \gamma - 1 \frac{d\rho}{\rho} \tag{5}$$

$$\frac{dP}{P} = \gamma \frac{d\rho}{\rho} \tag{6}$$

Suppose a nebula contracts till equilibrium is reached. Then the pressure gradient equals the gravitational potential:

$$\frac{dP}{dr} = -\rho g = -G\rho \frac{M}{r^2} \tag{7}$$

and for a system with a spherical symmetry

$$dP = -G\frac{4\pi}{3r^2}\rho^2 r^3 dr = -G\rho^2 r \frac{4\pi dM}{12\pi\rho r^2} \rightarrow P \propto \rho^{\frac{4}{3}}. \tag{8}$$

Any stable thermal pressure behaves as a polytrope of index $\delta \geq \frac{4}{3}$, while $\delta$ is smaller than $\frac{4}{3}$ in collapsing systems. In statistical mechanics, $q$ degrees of freedom imply that $\gamma = 1 + \frac{2}{q}$. Then if $q = 3$ $\gamma$ turns out to be $\frac{5}{3}$ while $q = 6$ leads to $\gamma = \frac{4}{3}$. Whenever $q$ is larger than 6 the structure is not stable. Furthermore, $\delta = \frac{n+1}{n}$, being $n$ the polytropic index and so $n = \frac{3}{2}$ implies $\delta = \frac{5}{3}$ and $n = 3$ $\delta = \frac{4}{3}$.

The above discussion holds for a perfect gas, whose pressure is only the pressure of the gas itself. Instead, in a stellar *plasma* the gas pressure is just a fraction $\beta$ of the total one and usually the radiation pressure is the remaining part $(1-\beta)$. For an adiabatic transformation of a plasma, admixture of gas and radiation, relations similar to the one valid for a prefect gas hold, but the exponent is no longer equal to a single $\gamma$ value and three adiabatic exponents $\Gamma_i$ have to be defined for transformations in presence of radiation:

$$\frac{dP}{P} = \Gamma_1 \frac{d\rho}{\rho} \tag{9}$$

$$\frac{dP}{P} = \frac{\Gamma_2}{\Gamma_2 - 1}\frac{dT}{T} \tag{10}$$

$$\frac{dT}{T} = (\Gamma_3 - 1)\frac{d\rho}{\rho} \tag{11}$$

Hence

$$\frac{dlnP}{ln\rho} = \Gamma_1 \tag{12}$$

$$\frac{dlnP}{dlnT} = \frac{\Gamma_2}{\Gamma_2 - 1} \tag{13}$$

$$\frac{dlnT}{dln\rho} = (\Gamma_3 - 1) \tag{14}$$

It can also be shown that when $\beta < 1$, also the molecular weight $\mu$ is variable and one can define its gradient:

$$\nabla_{\mu} = \frac{dlnT}{dln\mu} \tag{15}$$

Relations from 9 to 15 are useful in stellar modeling to define the borders between regions where radiation or convection dominate.

## 3 The Virial theorem

Let's consider an interstellar cloud made of many particles of masses $m_i$, in a reference frame where $r_i$ is the distance vector of each particle $m_i$ from the origin. Each particle is subject only to internal forces, namely gravity, and such a system is called a "virial" system (Clausius 1870 [3]). Its energy is defined as:

$$\sum_{i=1} m_i \frac{d^2 r_i}{dt^2} = -G \sum_{i=1} \frac{M m_i}{r_i^2} \tag{16}$$

By scalary multiplying by the vector $r_i$ one obtains:

$$\sum_{i=1} m_i r_i \ddot{r}_i - G \sum_{i=1} \frac{M m_i}{r_i} \equiv \Omega \tag{17}$$

By definition the moment of inertia and the kinetic energy of each particle are $I_i = m_i r_i^2$ and $E_k = \frac{1}{2} m_i \dot{r_i}^2$, respectively. The whole cloud moment of inertia is $I = \sum_{i=1} m_i r_i^2$, while the total kinetic energy is equal to the thermal energy $E_k = U \equiv \frac{3}{2} nkT$. By deriving $I$ twice the following relation is easily found:

$$\sum_{i=1} \frac{d^2 (m_i r_i^2)}{dt^2} = 2m\dot{r^2} + 2mr\ddot{r} \tag{18}$$

Hence

$$\frac{1}{2} \frac{d^2 I}{dt^2} = 2E_k + \Omega \tag{19}$$

where $m = \sum_{i=1} m_i$ and when the cloud reaches equilibrium:

$$E = \frac{1}{2}\Omega + \Omega = U - 2U \rightarrow E = -U \tag{20}$$

Any increasing $E$ results in a decrease in $U$ (and $T$) and self-gravitating clouds have a negative specific heat.

## 4 Evolutionary time scales

Equation 17 can be written for the unit volume so that $m = \rho$. Then the cloud starts heating due to contraction (equation 20), and a pressure gradient starts to appear (see [1]), so that the force balance becomes:

$$\rho \frac{\partial^2 r}{\partial t^2} + \frac{\partial P}{\partial t} = -\frac{G(M_r \rho)}{r^2} \tag{21}$$

At the very beginning the pressure gradient is minimal, and one can approximately assume: $\frac{dP}{dr} \simeq 0$, then the cloud starts to contract. Let us make the hypothesis, for the sake of simplicity, that the velocity (of each particle) is proportional to the distance from the centre ("homologous" collapse). Assuming an average density $\rho_0$ for the nebula its mass will be $M_r \equiv (M_r)_0 = \frac{4}{3}\pi \rho_0 r_0^3$: hence one can easily compute the the free fall time scale, namely the time required for the unimpeded collapse driven by gravity:

$$\tau_{ff} = \left( \frac{3\pi}{32 G \rho_0} \right)^{\frac{1}{2}} \rightarrow \tau_{ff} \simeq \frac{1}{\sqrt{G\rho}} \tag{22}$$

By assuming $M_r = M_\odot$ and $\rho = \rho_{\odot,average}$, the free fall time is about $1800s$, namely the Sun would evolve in only half an hour. Therefore it is evident that another source of energy is needed to feed stars and ensure a time scale of billions of years for their evolution. This was the starting point for the thermodynamical analysis by Kelvin and Helmoltz. According with them, the total energy of a star $E$ is constant, only half of the gravitational energy goes into keeping the gas warm and, if we have a prefect gas, with only thermal degrees of freedom, half gravitational energy can be radiated [4].

Now let's consider the gravitational potential of a gaseous sphere

$$\Omega = -\int_0^{M_\star} G \frac{M dM}{r_M} = -\frac{3}{5} G \frac{M_\star^2}{R} \tag{23}$$

being $R$ and $M_\star$ the stellar radius and mass, respectively. The energy that can be radiated is half of the value resulting from equation 23. If it is emitted at the present solar rate ($3.8 \times 10^{33} erg/sec$), then the Kelvin-Helmoltz time scale is:

$$\tau_{KH} = \frac{3GM_\odot^2}{10R_\odot L_\odot} \simeq 2 \cdot 10^7 yr \tag{24}$$

In 1920 Rutherford and Holmes published the results of several years of measurements on ancient rocks [5]. From the abundance of actinides they derived for the Earth an age larger than 1 Gyr. That was the proof that gravity cannot be the main source of energy for the Sun: it would be much younger than the Earth itself.

Firstly, sir Eddington [6] suggested that a sufficiently long lifetime would be guaranteed to the Sun by the fusion of 4 protons into a He nucleus. The binding energy of $^4$He is 26.71 MeV (the 0.7% of the mass of 4 protons) then the energy delivered by the Sun by burning about 10% of its H mass can be calculated starting from the Einstein's equation:

$$E = mc^2 \rightarrow \Delta E = 0.1 \cdot 1.989 \cdot 10^{33} \cdot 0.007 \cdot (2.99 \cdot 10^{10})^2 \simeq 1.3 \cdot 10^{51} erg \tag{25}$$

Being the present solar luminosity (namely the energy radiated per unit of time) $L_\odot = 3.8 \cdot 10^{33} erg/s$ the time scale for solar evolution powered by H-burning is

$$\tau_{nucl} = \frac{1.3 \cdot 10^{51}}{3.8 \cdot 10^{33}} = 3.4 \cdot 10^{17} s \rightarrow \tau_{nucl} \geq 10^{10} yr \tag{26}$$

Indeed, $4.603 \cdot 10^9 yr$ is the estimated age of the Sun, which is supposed to spend in total about $0.8 \cdot 10^{10} yr$ on the Main Sequence, burning H in its core. In 1928 Gamow [7] showed, by solving Schroedinger's equation in a simple exercise, that he Tunnel Effect allows the Sun to fuse protons and synthesize helium. In 1938 Bethe and Critchfield [8] published a detailed list of the nuclear reactions, which operate the transmutation of H into He in stellar environments, the so called pp-chain.

## 5 The Jeans' mass

Beside a too short stellar lifetime, another problem related to the Virial theorem is that it would allow the formation only of very massive objects. Applying equation 16 to the interstellar medium one obtains that the Jeans mass, namely the minimum mass needed to form a star (in equilibrium),is:

$$M_J = \left(\frac{5k}{G\mu m_H}\right)^{\frac{3}{2}} \left(\frac{3}{4\pi\mu m_H}\right)^{\frac{1}{2}} \frac{T^{\frac{3}{2}}}{n^{\frac{1}{2}}} \simeq 2 \cdot 10^{35} \frac{T^{\frac{3}{2}}}{n^{\frac{1}{2}}} \tag{27}$$

By assuming $T = 10K$ and a density $n$ of 10 particle per $cm^3$, $M_J$ is about $2 \cdot 10^{36} g \simeq 1000 M_\odot$, which is the order of magnitude of the masses of open clusters and not of a single star. According to equation 27 massive objects of 20-30$M_\odot$ may form in the compact cores of interstellar medium clouds, where $T \sim 20 - 30K$ and $n \geq 10^8 cm^{-3}$. However, the smaller is the mass the larger is the number of stars with that mass in a galaxy, then how could the majority of the stars form? They form by fragmentation triggered by forces external to the nebula. The cloud cores actually form by a slow accretion induced by extra pressures (galactic dynamics, magnetic fields, etc...). In these case equation 20 is:

$$2U + \Omega = 3 \int PdV \tag{28}$$

where $3 \int PdV$ is the external work spent in fragmentation. According to the Virial theorem, when the cloud is in thermal equilibrium half of its gravitational energy can br used to break solids and molecules (non thermal degrees of freedom) and when the density $\rho$ increases, $M_J$ rapidly decreases allowing the formation of typical stellar masses. Every perturbation of $\rho$ induces fragmentation and smaller mass clusters form. If these fragments reach the virial equilibrium they can be described by polytropes with $\delta \geq 4/3$. For them:

$$T \propto \rho^{\delta-1} \rightarrow M_J \propto \rho^{\frac{3\delta-4}{2}} \rightarrow M_J \propto \rho^{\frac{1}{2}} \tag{29}$$

The denser the fragments the more massive the formed stars are.

## 6 Radiative transport

When energy is transported by the interaction of photons with matter (photons being absorbed and re-emitted by each stellar layer, with a very short mean free path) the fraction of the flux $d\Phi_\nu$ across a layer $dr$ will be proportional to the flux $\Phi_\nu$ itself.

$$d\Phi_\nu = -\Phi_\nu k_\nu \rho dr \rightarrow \frac{d\Phi_\nu}{dr} = -k_\nu \rho \Phi_\nu \tag{30}$$

where the constant $k_\nu$ deals with interactions between radiation and matter per unit volume. Since $dP_r = \frac{d\Phi_\nu}{c}$ and $\Phi_\nu = \frac{L}{4\pi r^2}$:

$$\frac{dP_r}{dr} = -\frac{k_\nu \rho L(r)}{4\pi c r^2} \tag{31}$$

## 7 The molecular weight

In a unit volume (whose mass is $\rho$) the number of particles is equal to the mass of the gas divided by the average mass of each particle:

$$n = \frac{\rho\mu}{m_H} \tag{32}$$

Stellar composition is often simply defined by the parameters X, Y and Z, being X is the mass fraction of hydrogen, Y the mass fraction of helium, and Z the metallicity, namely the mass fraction of the heavier elements ($A > 4$). Thus $X + Y + Z \sim 1$, but in a stellar plasma the particles include ions and free electrons and every fully ionized hydrogen atom releases one electron, hence the particles associated to H are:

$$n_H = 2X\frac{\rho}{m_H} \tag{33}$$

Every He ion has mass 4 $m_H$ and delivers 2 electrons, whose masses can be neglected being much smaller than the nuclear one, but the number of particles is 3, then

$$n_{He} = \frac{3}{4}Y\frac{\rho}{m_H} \tag{34}$$

Finally for heavier elements (the so-called metals), the most abundant ones are those made of $\alpha$ particles ($^{12}$C, $^{16}$O, $^{20}$Ne...), for which the charge is half the atomic mass ($A/2$). Hence for any nucleus of mass $A_i$:

$$n_{A>4} = \frac{A}{2}\cdot\frac{1}{A}Z\frac{\rho}{m_H} = \frac{Z}{2}\frac{\rho}{m_H} \tag{35}$$

The average molecular weight $\mu$ for a stellar plasma can then be computed combining equation 33, 34, and 35:

$$\mu = \frac{\rho}{m_h n} = \left[2X + \frac{3}{2}Y + \frac{1}{2}Z\right]^{-1} \tag{36}$$

Values of the average molecular weight vary depending on the stellar environment under analysis. For a pure hydrogen gas ($X = 1$) $\mu = 0.5$, while for a mixture of pure heavy elements $\mu = 2$, as for example in the innermost cores of massive stars before the core-collapse of a supernova. A ionized gas with the solar composition, where the total abundance of elements heavier than 4 is the solar metallicity Z=0.015 (X = 0.71 and Y = 0.275) $\mu \simeq 0.612$. For electrons we have one particle per H nucleus, one-half per He nucleus and many electrons per ions of heavy elements. Hence:

$$\mu_e \simeq \left[X + \frac{Y}{2} + \frac{1}{2}(1 - X - Y)\right]^{-1} \tag{37}$$

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## 8 Degeneracy

Degeneracy prevents stars or protostars to evolve. Indeed whenever the plasma is degenerate the on-set of the nuclear burning typical of the next evolutionary stage is hampered. It is so for clouds with mass smaller than $0.08M_\odot$, which will never reach the physical condition needed to burn H in their core, or for red giants whose He core is smaller than $1.4M_\odot$ (which is allowed to burn just by a late instability called He-flash), or for any other evolved object that is not massive enough to promote a further nuclear burning in its core (e.g. C-burning after the He-burning, O-burning after C-burning, etc...).

In a simplified way one can state that the inner layers of a star (or a cloud) become degenerate when the temperature does not grow enough to allow the fusion between charged particles, being its mass too low. As already explained, when gravity dominates the density increases, because only half gravitational potential energy is lost through radiation and no nuclear energy is provided to compensate the rest. This is so as during the pre-main sequence or at the end of each nuclear burning phase along stellare evolution. Since the stellar plasma is not made of ideal gases, under those conditions particles become in contact with one another, each one occupying a volume $l^3$ where $l$ is the mean free path $l \sim \sqrt[3]{n^{-1}} = \sqrt[3]{m/\rho}$. The length $l$ can be easily estimated from the De Broglie wave length $\lambda_{DB} = h/mv$, here $h$ is the Plank constant and $v$ the particle thermal velocity in a Maxwell distribution, $v \simeq \sqrt{3kT/m}$. By some simple algebra the critical density above which the system will be degenerate turns out to be:

$$\rho_{deg} \propto \sqrt{\frac{T^3}{m^5}} \tag{38}$$

For $T$ values typical of stars that have just left the Main Sequence (a few $10^7 K$) and for particles of mass equal to the mass of electrons, the critical density is about $10^6 g/cm^3$. Stars, which encounter electron degeneracy before settling on the Main Sequence, evolve at very high central density and radiate their gravitational energy with no H-burning. Such objects are called brown dwarfs (and giant planets as Jupiter are at the lowest). For particles more massive than electrons $\rho_{deg}$ is larger: electrons become a degenerate gas at a density much lower than the ones at which protons or neutrons do , typically by a factor $2000^{5/2}$, according to equation 38.

## 9 The Eddington Limit

In the same way the degeneracy fixes the lower limit of the stellar masses the luminosity fixes its upper limit, called the Eddington limit [9]. Indeed, radiation pressure on electrons and ions at a stellar surface is proportional to its luminosity and it drives stellar winds. When radiation pressure exceeds the gravitational one the star approaches the Eddington luminosity $L$ where radiation pressure would destroy it.

$$L = \frac{4\pi G c m_p M}{\sigma_e} = 1.26 \times 10^{38} \frac{M}{M_\odot} erg \cdot s^{-1} \tag{39}$$

Equation 39 gives the maximum luminosity that an object of mass M emitting its radiation isotropically can have, above this limit the star would be blown away by its own radiation pressure. Therefore, by inverting the formula, one obtains the mass upper limit for a star with a given luminosity $L$:

$$M_E = 8 \times 10^5 \frac{L}{10^{44} erg \cdot s^{-1}} M_\odot \tag{40}$$

which corresponds to a limiting mass slightly lower than 200 $M_\odot$.

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