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Conservation Laws and Hydrodynamics

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Abstract. Recent years have seen much development in analyzing the structure of relativistic hydrodynamics. In this proceeding, some of the developments are highlighted including issues related to pseudo-gauge transformations and spin hydrodynamics.

1 Introduction

Hydrodynamics has a long history in describing heavy ion collisions starting with Landau's seminal paper [1] and Bjorken's [2]. In the era of RHIC and the LHC, hydrodynamics has become an indispensable tool to describe the bulk dynamics of relativistic heavy ion collisions.

In this 2022 Strangeness in Quark Matter conference, many of the recent developments in hydrodynamics were reported. On spin hydrodynamics and Λ polarization, Palermo [3], Liao [4] and Buzzegoli [5] presented their work. On the subject of nonolinear causality conditions, Plumberg [6] presented his and collaborators' work. On the development of hydro initial states, Shen [7] presented recent developments for ultra-peripheral collisions and Kanakubo [8] presented her recent work on the core-corona model. On applying hydrodynamics to systems near the QCD critical point, Pradeep [9], Wu [10], and Sogabe [11] presented their work in this conference.

As this is a brief introduction to the recent developments, inevitably only a limited number of topics can be discussed. Below, I will mainly focus on the pseudo-gauge transformation and its relevance in developing spin hydrodynamics. I should emphasize that this is a personal choice dictated by my familiarity with the subjects [12].

2 Is any current density observable?

Hydrodynamics starts with conservation laws

$$\partial_\mu J^{\mu,q} = 0 \tag{1}$$

where q is the index for different conserved quantities such as energy, momentum, electric charge and baryon number. Mathematically, a conserved current density is *not* uniquely defined. One can always add a "pseudo-gauge" term

$$J'^{\mu,q} = J^{\mu,q} + \partial_\lambda B^{\lambda\mu,q} \tag{2}$$

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where $B^{\lambda\mu,q} = -B^{\mu\lambda,q}$ is the pseudo-gauge potential. Adding $\partial_\lambda B^{\lambda\mu,q}$ neither changes the conservation law $\partial_\mu J'^{\mu,q} = 0$, nor the total charge provided that $B^{i0,q}$ vanishes at the boundary of the volume. But it does change the local current density since $J^{\mu,q}(x) \neq J'^{\mu,q}(x)$.

Does this mean that a current density is not observable? That depends on whether there exist other constraints that may select a particular form to be physical. For instance, in the classical particle dynamics, a point particle carries well-defined energy, momentum and other conserved charges. Hence, the classical current density

$$J^{\mu,q}_{\rm cl}(x) = \sum_{s=\text{ptcles}} \frac{q_s}{p^0_s} p^\mu_s \delta^{(3)}(\mathbf{x} - \mathbf{x}_s(t)) \tag{3}$$

is physical and observable as long as a particle is observable. Adding an arbitrary $\partial_\lambda B^{\lambda\mu,q}$ to $J^{\mu,q}_{\rm cl}$ is still a possibility, but the fact that a point particle carries definite amounts of energy, momentum and other charges clearly selects Eq.(3) to be the physical current density.

In reality, due to the uncertainty principle, particles cannot be localized as in Eq.(3). Particle positions are inevitably smeared so that

$$J^{\mu,q}_C(x) = \sum_{s=\text{ptcles}} \frac{q_s}{p^0_s} p^\mu_s g(\mathbf{x} - \mathbf{x}_s(t)) \tag{4}$$

with the normalization condition $\int d^3x\, g(\mathbf{x} - \mathbf{x}_s(t)) = 1$ that preserves the total charge. This expression still satisfies $\partial_\mu J^{\mu,q}_C = 0$ as long as $d\mathbf{x}_s/dt = \mathbf{p}_s/p^0_s$, but since there can be many choices for the smearing function, this does not define a unique value of the current density at x. For any change $\delta J^{\mu,q}_C$ introduced by a change of the smearing function, one can always find $B^{\lambda\mu,q}$ that satisfies

$$\delta J^{\mu,q}_C = \partial_\lambda B^{\lambda\mu,q} \tag{5}$$

since it has the same form as the Maxwell equation. Hence, without any further constraint on the form of the smearing function, the continuous current density $J^{\mu,q}_C$ can be ambiguous.

I have been so far careful to always refer to $J^{\mu,q}$ as a current density. This is to distinguish $J^{\mu,q}$ from the current. An electric current, for instance, is defined as

$$I_S = \int_S d\mathbf{S} \cdot \mathbf{J}^e \tag{6}$$

where $\mathbf{S}$ is the surface through which the electric current is measured and $\mathbf{J}^e$ is the spatial part of the electric current density. The current itself is a scalar quantity, not a vector quantity. One can easily show that as long as $\mathbf{B}$ vanishes at the boundary of the surface, I_S remains unchanged.

Does this mean that no current density is actually observable due to the pseudo-gauge freedom? In order to answer this question, consider the electric current density J^μ_e. As discussed above, the conservation law is not affected when a pseudo-gauge potential with the property $B^{\nu\mu}_e = -B^{\mu\nu}_e$ is added to J^μ_e. Using the new current density, The Maxwell equation becomes

$$\partial_\nu F'^{\nu\mu} = J'^\mu_e = J^\mu_e + \partial_\nu B^{\nu\mu}_e \tag{7}$$

The electric and magnetic field generated by J'^μ_e is then

$$F'^{\nu\mu} = F^{\nu\mu} + B^{\nu\mu}_e \tag{8}$$

where $F^{\nu\mu}$ is generated by J^μ_e. But this doesn't make much sense. The electric field and the magnetic field are measurable observables. They shouldn't be ambiguous. Hence, there must be a *physical* electric current density that generates the *measured* electric and magnetic fields.

For the energy-momentum tensor, a similar situation arises with the Einstein's equation where a symmetric Belinfante energy-momentum tensor (see below) sources the gravitational field. Since gravitational field is observable and measurable, the Belinfante tensor must be the physical energy-momentum tensor.

There are two lessons I would take from the discussions above. One is that as long as a current density generates an observable and measurable field (such as the electric and magnetic fields), one should be able to identify the unique physical and observable current density. The second lesson is that even though unambiguously identifying *classical* current density using the above arguement may be possible, identifying quantum current density could be still ambiguous (c.f. Eq.(5)). However, this ambiguity should represent only the $O(\hbar)$ correction to the classical current density.

3 Stress-energy tensors with spin

Hydrodynamics primarily deals with the energy-momentum tensor. Consider a field-theory Lagrangian $L(\varphi^A, \partial_\mu\varphi^A)$ where A is the field index. Using Noether's theorem, one can show that the conserved current densities for the translational symmetry in space-time is given by

$$\Theta^{\mu\nu} = \Pi^\mu_A \partial^\nu \varphi^A - g^{\mu\nu} L(\varphi_A, \partial_\mu\varphi^A) \tag{9}$$

where $\Pi^\mu_A = \partial L/\partial(\partial_\mu\varphi_A)$. This tensor satisfies

$$\partial_\mu \Theta^{\mu\nu} = \partial^\nu \varphi_A \left(\partial_\mu \Pi^\mu_A - \frac{\partial L}{\partial \varphi_A} \right) \tag{10}$$

Hence if φ_A satisfies the Euler-Lagrange equation $\partial_\mu \Pi^\mu_A = \partial L/\partial\varphi_A$, then $\Theta^{\mu\nu}$ is a conserved energy-momentum current density. The first index μ is the space-time index for the current density components and the second index ν indicates what is being conserved. Namely, $\nu = 0$ is the label for the energy current and $\nu = 1, 2, 3$ are for the momentum currents playing the role of the index q in the previous section. This canonical energy-momentum tensor is not symmetric unless it only contains scalar (spin $S = 0$) particles, and it is not gauge invariant when applied to gauge theories. As such, it cannot yet be directly interpreted as the measurable energy-momentum current density.

To see the consequences of having a non-symmetric $\Theta^{\mu\nu}$, define the orbital angular momentum current density

$$\mathcal{L}_O^{\lambda\mu\nu} = x^\mu \Theta^{\lambda\nu} - x^\nu \Theta^{\lambda\mu} \tag{11}$$

This tensor is in general not conserved $\partial_\lambda \mathcal{L}_O^{\lambda\mu\nu} = \Theta^{\mu\nu} - \Theta^{\nu\mu}$ unless all the fields have $S = 0$. If fields have non-zero spins, then the total angular momentum current density must include the spin angular momentum part as well as the orbital angular momentum part

$$\mathcal{L}_{\rm tot}^{\lambda\mu\nu} = x^\mu \Theta^{\lambda\nu} - x^\nu \Theta^{\lambda\mu} + \mathcal{S}^{\lambda\mu\nu} \tag{12}$$

where $\mathcal{S}^{\lambda\mu\nu}$ is the spin current density. The statement of the total angular momentum conservation becomes

$$\partial_\lambda \mathcal{S}^{\lambda\mu\nu} + 2\Theta^{\mu\nu}_{(a)} = 0 \tag{13}$$

In other words, the anti-symmetric part of the energy-momentum tensor, $\Theta^{\mu\nu}_{(a)} = (1/2)(\Theta^{\mu\nu} - \Theta^{\nu\mu})$, encodes the spin angular momentum part of the energy and momentum. That part is then necessarily of $O(\hbar)$ since the spin current density is $O(\hbar)$.

To find the pseudo-gauge transformation that results in the physical energy-momentum tensor, let

$$T^{\mu\nu} = \Theta^{\mu\nu} + \partial_\lambda B^{\lambda\mu,\nu} \tag{14}$$

where $T^{\mu\nu}$ is the symmetric tensor that we are after. Using $T^{\mu\nu} - T^{\nu\mu} = 0$, we get

$$0 = \Theta^{\mu\nu} - \Theta^{\nu\mu} + \partial_\lambda(B^{\lambda\mu,\nu} - B^{\lambda\nu,\mu}) \tag{15}$$

Using the conservation of total angular momentum, Eq.(13), one can identify

$$\mathcal{S}^{\lambda\mu\nu} = B^{\lambda\mu,\nu} - B^{\lambda\nu,\mu} \tag{16}$$

Equivalently,

$$B^{\lambda\mu,\nu} = \frac{1}{2}\left(\mathcal{S}^{\lambda\mu\nu} - \mathcal{S}^{\mu\lambda\nu} - \mathcal{S}^{\nu\lambda\mu}\right) \tag{17}$$

that yields

$$T^{\mu\nu} = \Theta^{\mu\nu}_{(s)} - \frac{1}{2}\partial_\lambda\left(\mathcal{S}^{\mu\lambda\nu} + \mathcal{S}^{\nu\lambda\mu}\right) \tag{18}$$

after explicit symmetrization. This form of the energy-momentum tensor is usually referred to as the Belinfante energy-momentum tensor and it can be shown that this is identical to the gravitational source term in the Einstein equation obtained by varying the space-time metric. As such, classically, this is the observable energy-momentum tensor. One should note that this form of energy-momentum tensor now incorporates the conservation of total angular momentum since $\partial_\mu T^{\mu\nu} = 0$ is satiafied only after using both the equations of motion and the total angular momentum conservation, Eq.(13). It also means that the energy density obtained from $T^{\mu\nu}$ contains the energy due to the spins [14].

The the total angular momentum current density is now just given by

$$\mathcal{L}^{\lambda\mu\nu}_{\rm tot} = x^\mu T^{\lambda\nu} - x^\nu T^{\lambda\mu} \tag{19}$$

which could be further divided this into the orbital angular momentum term and the spin term although this division is not so clear-cut.[1] This also means that there is no separate spin current density which presents a bit of a problem when formulating spin hydrodynamics.

4 Spin hydrodynamics

STAR has measured Λ polarizations that showed a surprising azimuthal pattern [15] that may indicate a strong spin-orbit coupling between the polarization of a hadron and the underlying fluid motion. In order to figure out where such a strong correlation can originate from, it is clear that we need a formulation of hydrodynamics that includes the spin degrees of freedom. This is, however, not straightforward: If one insists that the energy-momentum tensor must be observable, then it must be the Belinfante tensor. But that means there is no separate spin current density. On the other hand, one can naturally get the spin current density via the anti-symmetric part of the canonical energy-momentum tensor. However, the canonical energy-momentum tensor is not physical.

There are currently two broad approaches to this problem. One is to approach the spin hydrodynamics from the semi-classical kinetic theory point of view [3, 5, 12, 16]. Another

[1]For instance, for a spin-1 gauge field, only the sum is gauge-invariant.

approach is to start with the general expression of the non-symmetric energy-momentum tensor and the spin current density tensor and develope a gradient expansion [14, 17]. One can then use the extended thermodynamic identities and the production of entropy to obtain the first order and the second order form of spin hydrodynamics. The relationships between various formulations of the spin hydrodynamics, however, is not yet fully settled and full spin hydrodynamics simulations are yet to be performed.

In all these approaches, a common task is to decompose the non-symmetric tensor into the physical symmetric part and the $O(\hbar)$ spin dependent part. This decomposition is necessary because quantities such as the energy density and the flow velocity must be defined to carry out any hydrodynamic calculations. In some sense, this largely deals with the pseudo-gauge problem. Namely, once we have identified the classical energy-momentum tensor (corresponding to the Belinfonte tensor), then only $O(\hbar)$ pseudo-gauge potentials are allowed in order not to spoil the classial part of the energy-momentum tensor.

5 Outlook

In this short review, I have tried to highlight some of the issues in formulating physical hydrodynamics that were raised in recent literatures. The two related topics I have chosen to highlight here, namely the issue of pseudo-gauge transformations and spin hydrodynamics, have been studied by several groups motivated by the Λ polarizations measured by the STAR collaboration at RHIC. Through such efforts, it has become clearer what role pseudo-gauge transformations can and cannot play in hydrodynamics in general and in spin hydrodynamics in particular.

Theoretically, the STAR Λ polarization measurement presents an interesting challenge. Although some resolutions of this puzzle[2] have been suggested [3], more studies, especially realistic spin hydrodynamics simulations, are needed to firmly settle it.

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