Spatiotemporal optical vortices: advances and mysteries - INVITED

Miguel A. Porras$^{1,*}$

$^1$Grupo de Sistemas Complejos, ETSIME, Universidad Politécnica de Madrid, Rios Rosas 21, 28003 Madrid, Spain

Abstract. The amount of transverse orbital angular momentum (OAM) carried by the spatiotemporal optical vortices (STOVs) is being hotly debated. In this contribution I unveil the mystery of the amount of total, intrinsic and extrinsic transverse OAM carried by STOVs. They do not carry any total transverse OAM about a static transverse axis crossing the STOV center. Yet, STOVs carry a transverse OAM about a moving transverse axis crossing the STOV center permanently, which is identified with the intrinsic OAM, and an opposite extrinsic transverse OAM about the fixed axis. These results raise questions about the interpretation of second-harmonic and high-harmonic experiments or simulations with STOVs and the capability of STOVs to transmit OAM to particles. Other advances such as observation of vortex splitting and topological charge flipping of high-order STOVs in free space and in dispersive media, and the transverse torque exerted by optical elements will be discussed.

1 Introduction

Spatiotemporal optical vortices (STOVs) are non-monochromatic wave packets that feature a phase line singularity perpendicular to the propagation direction, in contrast to the standard longitudinal vortices in monochromatic light beams [1]. Their experimental realization [2,3] promises new forms of interactions with matter following analogous lines to the broad applications of standard vortices. However, the properties of STOVs are not yet clear, in particular, the amount of transverse orbital angular momentum (OAM) remains a mystery. Different authors provide different values for the same STOV [4-7].

Here we evaluate the total, intrinsic and extrinsic transverse OAM of paraxial and quasi-monochromatic (many cycle) pulsed beams, and apply them to STOVs. We strictly adhere to classical electromagnetism, without introducing angular momentum operators [4] or photon wave functions [5-7]. The angular momentum carried by a wave packet is identified with the angular momentum flux [8] integrated over a transversal section, say $z = \text{const.}$, and integrated in time. For linearly polarized wave packets, this angular momentum is OAM, whose transversal component, e.g., the $y$ component, gives the transverse OAM.

When applied to STOVs in free space, we find that, the total transverse OAM about a fixed transverse axis passing through the STOV center vanishes. The transverse OAM per unit energy ("per photon") about a moving transverse axis crossing permanently the STOV center (moving at $c$), i.e., the intrinsic part, is $l \gamma/2\omega_0$, where $l$ is the STOV topological charge, $\omega_0$ is the carrier frequency and $\gamma$ is the ellipticity of the vortex at the plane where it is elliptical. For circular STOVs ($\gamma = 1$) this is half the OAM content of longitudinal vortices.

In addition, advances such as understanding the complex behavior of the singularities and their topological charges in STOVs will be discussed.

2 Transverse OAM

Assuming propagation along the positive $z$ direction, that the phase line singularity is parallel to the $y$ axis, and that the STOV center moving at velocity $c$ is permanently at $x = 0$, it is relevant to evaluate the $y$-component of the angular momentum about the $(x,z) = 0$ axis as the fixed axis. The OAM is evaluated from

$$ J_y = \iiint M_{yx}dxdydt, \quad (1) $$

where the integral extends to the whole transversal plane and to all times. $M_{yx}$ is the the flux density of the $y$ component of the angular momentum through a surface perpendicular to $z$, has units of angular momentum per unit time and unit area [8], and is given by

$$ M_{yx} = zT_{xz} - xT_{zz}, \quad (2) $$

Here, $T_{xz}$ and $T_{zz}$ are similar flux densities for the momentum, the former for the $x$ component, and the latter for the $z$ component, both through surfaces perpendicular to the $z$ direction [8], and are given by

$$ T_{xz} = - \varepsilon_0 E_x E_z - \mu_0^{-1} B_x B_z, \quad (3) $$

$$ T_{zz} = \frac{1}{2} \left[ \varepsilon_0 (E_x^2 + E_y^2 - E_z^2) + \mu_0^{-1} (B_x^2 + B_y^2 - B_z^2) \right], \quad (4) $$

where $E_i$ and $B_i$ are the electric and magnetic fields. The total flux through a transversal section and at all times in Eq. (1) is naturally identified with the carried transverse OAM. Similarly, the energy transported by the wave packet is given by $W = \iint S_zdxdydt$, where $S_z = \mu_0^{-1} (E_z B_y - E_y B_z)$ is the energy flux density.

3 Quasi-monochromatic, paraxial fields

Starting with scalar, pulsed beam solutions $\psi$ of the Schrödinger equation for paraxial and many-cycle fields, $\partial_x \psi = i\Delta_+ \psi/2\kappa_0$ (in absence of dispersion), where $\kappa_0 = \omega_0/c$ and $\Delta_+ = \partial_x^2 + \partial_y^2$, electric and magnetic fields can be constructed according to Lax’s method as
\[ E_x = \text{Re}(\psi e^{-i\omega t}), \quad E_y = \text{Re}(i\partial_\tau \psi e^{-i\omega t}/k_0), \quad B_y = \text{Re}(\psi e^{-i\omega t}/c), \quad B_x = \text{Re}(i\partial_\tau \psi e^{-i\omega t}/k_0c), \quad \text{and} \quad E_y = B_x = 0 \text{ for linear polarization along } x, \text{ where } t' = t - z/c \text{ is the local time. For linear polarization along } y, \text{ replace } x \leftrightarrow y \text{ in the above expressions.} \]

For these fields, the transverse OAM can be evaluated as
\[ J_y = \iiint (M_{yx}) dx dy dt, \text{ where } (M_{yx}) = \xi(T_{xx}) - x (T_{yx}) - x (T_{xz}) + x (T_{yz}) \text{ and the brackets stand for cycle-average. From Lax's fields, one obtains} \]
\[ \langle T_{xx} \rangle = \frac{e_0}{2k_0} \text{Im}(\psi^* \partial_\tau \psi), \quad (T_{yx}) = \frac{1}{2} e_0 |\psi|^2. \tag{5} \]

Similarly, the energy is given by
\[ W = \iiint (\mathcal{S}_y) dx dy dt, \text{ where } (\mathcal{S}_y) = \frac{1}{2} e_0 |\psi|^2, \text{ i.e.,} \]
\[ W = \frac{1}{2} e_0 c \iiint |\psi|^2 dx dy dt. \tag{6} \]

The above expressions yield the total transverse OAM about the fixed transverse \( y \) axis \((x,z) = 0\). Since STOVs move at speed \( c \), the transverse OAM about the moving transverse axis \((x,z - ct) = 0\) passing permanently through the STOV center, will analogously be given by
\[ f_y^{(0)} = \iiint (M_{yx})^{(0)} dx dy dt, \text{ where } (M_{yx})^{(0)} = (x - ct)(T_{xx}) - x (T_{yx}) - x (T_{xz}) + x (T_{yz}), \text{ which is identified with the intrinsic transverse OAM, and} \]
\[ f_y = J_y - f_y^{(0)} \text{ with the extrinsic OAM.} \]

Using all above equations, the results for the total and intrinsic transverse OAM are obtained to be
\[ J_y = \frac{e_0 z}{2 k_0} \iiint \text{Im}(\psi^* \partial_\tau \psi) dx dy dt', \]
\[ - \frac{1}{2} e_0 \iiint |\psi|^2 x dx dy dt', \tag{7} \]
\[ f_y^{(0)} = - \frac{ec}{2k_0} \iiint \text{Im}(\psi^* \partial_\tau \psi) t' dx dy dt', \tag{8} \]
where \( t \) is conveniently replaced with \( t' + z/c \), and we have used that the integral in \( t \) to all times is the same as the integral in \( t' \) to all local times.

### 4 Spatiotemporal optical vortices

It can be demonstrated that \( J_y, f_y^{(0)} \), are conserved on propagation, and obviously \( W \) as well. We take advantage of this fact to evaluate these physical quantities at the plane where the STOV has elliptical symmetry.

At this plane, STOVs are of the form \( \psi = f(\rho)e^{-i\omega t} \), where \( \rho = \sqrt{x^2 + \tau^2} \), and \( \varphi = \arctan \frac{\xi}{t}, \) with \( \tau = t'/t_0 \) and \( \xi = x/x_0 \), are polar coordinates in the spatiotemporal plane \( t'/t_0 - x/x_0 \). The parameters \( x_0 \) and \( t_0 \) determine the STOV ellipticity in the \( z-x \) plane as \( y = c t_0 / x_0 \). Note that, since \( t' = t - z/c \), positive \( l \) corresponds to positive topological charge, i.e., the phase increases counterclockwise in the \( z-x \) plane. Note also that an arbitrary profile \( g(y) \) along the \( y \) direction should multiply \( \psi \), but this only leads to factorized integrals in (6), (7) and (8) that cancel when evaluating the transverse OAM per photon. When \( \psi = f(\rho)e^{-il\tau} \) is introduced in (6), (7) and (8), and the integrals in \( x \) and \( t' \) are evaluated using the above polar coordinates, the results for the total, intrinsic and extrinsic transverse OAM per photon, i.e., per unit energy, are
\[ J_y / W = 0, \quad f_y^{(0)} / W = \frac{1}{2} \omega, \quad f_y / W = -\frac{1}{2} \omega. \tag{9} \]

Thus, the total transverse OAM of STOVs vanishes. The intrinsic transverse OAM of round STOVs \((\gamma=1)\) is half the OAM of standard vortices in monochromatic light beams. Since the total OAM vanishes, the extrinsic is opposite to the intrinsic.

### 5 Conclusions

It should be noted that these results disagree with those reported in [4-7]. The differences with [4-7] may arise from the use of different formalisms, in particular, angular momentum operators are used in [4]. The discrepancy with [5-7] may arise from the use of different STOV "centroids". Here the centroid of the intensity \((x,z - ct) = 0\) is chosen, while in [5-7] the centroid of the photon wave function is chosen, which is slightly shifted from \( x = 0 \).

Other issues such as the sophisticated propagation properties of high-order STOVs in free space and dispersive media, and the torque exerted by optical elements will be discussed in this talk.

This work has been partially supported by the Spanish Ministry of Science and Innovation, Gobierno de España, under Contract No. PID2021-122711NB-C21. The author also acknowledges support as visiting professor of La Sapienza University and Dipartimento di Fisica of La Sapienza.

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