Quarteting in deformed $N = Z$ nuclei

M. Sambataro\textsuperscript{1,*} and N. Sandulescu\textsuperscript{2,**}

\textsuperscript{1}Istituto Nazionale di Fisica Nucleare - Sezione di Catania, Via S. Sofia 64, I-95123 Catania, Italy
\textsuperscript{2}National Institute of Physics and Nuclear Engineering, P.O. Box MG-6, Magurele, Bucharest, Romania

Abstract. We describe deformed $N = Z$ nuclei in a formalism of $\alpha$-like quartets. Quartets are constructed variationally by resorting to the use of proper intrinsic states. Various types of intrinsic states are introduced which generate different sets of quartets for a given nucleus. Energy spectra are generated via configuration-iteration calculations in the spaces built with these quartets. The approach is applied to $^{25}$Mg and $^{29}$Si in the $sd$ shell and to $^{48}$Cr in the $pf$ shell. In all cases a good description of the low-lying spectra is achieved. As a peculiarity of the approach, a close correspondence is observed between the various sets of quartets employed and the occurrence of well defined band-like structures in the spectra of the systems under study.

1 Introduction

The study of $N = Z$ nuclei is one of the key issues of contemporary nuclear structure physics. $N = Z$ nuclei are characterized by having an equal number of protons and neutrons sharing the same orbitals. Owing to this fact, the proton-neutron ($pn$) interaction is expected to play a relevant role in these nuclei.

Understanding the type of correlations induced by the $pn$ pairing force in the wave function of $N = Z$ nuclei has proved to be a not trivial task [1–3]. An unambiguous indication of the type of correlations that are generated by the $pn$ pairing in $N = Z$ nuclei is provided by the exact analytical solutions of realistic Hamiltonians. In the case of the isovector pairing, finding such solutions has gone through various works spread over several decades [4–9], the first complete description of both eigenvalues and eigenstates being provided in 2006 by Dukelsky et al. [7]. However, the complicated mathematical structure of these exact solutions has hindered a simple understanding of the underlying correlations.

In a recent work [10], we have provided an alternative and more transparent description of the exact solutions of the isovector pairing which has evidenced a peculiar aspect of these solutions which had escaped the previous investigations. Indeed, we have shown on an analytic basis that what characterizes the isovector pairing Hamiltonian in $N = Z$ systems is the occurrence in its eigenstates of correlated four-body $\alpha$-like structures made by two protons and two neutrons coupled to $T = 0$ (“quartets”).

One needs to remark that, well before the exact treatment just discussed the essential role played by four-body correlations in $N = Z$ systems subject to an isovector pairing force, but limited to the special case of degenerate single-particle levels (the so-called $SO(5)$ model), had already been emphasized in Ref. [11]. More generally, quartets have definitely a long history in nuclear structure physics [12–19], but their complexity has undoubtedly represented a hindrance to the development of quartet models.

In a recent past, the spectra of $N = Z$ nuclei have been constructed in terms of $T = 0$ quartets with various angular momenta [20–22]. In this approach, hereafter referred to as Quartet Model (QM), spectra of both positive and negative parity states have been generated by carrying out configuration-interaction calculations in a space of states formulated as products of these quartets. A crucial aspect of these calculations has consisted in the definition of the quartets to involve in the calculations. In these early works, we have adopted the criterion of assuming as $T = 0$ quartets those defining the low-lying eigenstates of the nearest $T = 0$ one-quartet systems [20–22]. While having the advantage of being straightforward, this “static” definition of the quartets is clearly not the most appropriate one since it fully neglects the effect of the Pauli principle on the amplitudes of the quartets when two or more of these quartets have to coexist in the same nucleus. Finding the most appropriate quartets to be employed in a QM calculation is a matter of primary importance for the validity of this approach and this has represented one of the goals of the present work.

The approach that we illustrate in this paper is based on the use of “dynamical” quartets, namely quartets which are fixed in correspondence with each nucleus [23]. Quartets result from the minimization of special intrinsic states. We present three different applications of our approach, two referring to the $sd$ shell ($^{25}$Mg and $^{28}$Si) and one to the $pf$ shell ($^{48}$Cr). We show that the new criterion to fix the quartets provides not only a good description of the low-lying states of these nuclei but, as a peculiarity, it also allows the identification of band-like structures in their spectra.
In Section 2, we describe our approach and present the applications. In Section 3, we summarize the results and draw the conclusions.

2 Formalism and applications

The quartets that we employ are formed by two neutrons and two protons coupled to total isospin \( T = 0 \) and angular momentum \( J \). The associated creation operator is defined as

\[
q_{JM}^+ = \sum_{i_1,j_1,i_2,j_2} q_{i_1,j_1,i_2,j_2,J,T}[a_{i_1}^* a_{j_1}^*]^J i_{T=0}^T[J_{JM}^+],
\]

where \( q_{i_1,j_1,i_2,j_2} \) creates either a proton or a neutron (depending on the isospin projection \( r \)) on the spherically-symmetric state \( i \equiv \{ n, l, j \} \). No restrictions on the intermediate couplings \( J_1T' \) and \( J_2T' \) are introduced and the amplitudes \( q_{i_1,j_1,i_2,j_2,J,T} \) are supposed to guarantee the normalization of the operator.

In the representation spanned by the quartets (1) we introduce a set of intrinsic states, the simplest of which has the form [23]

\[
|\Theta_q\rangle = N_q(Q_q^+)^n|0\rangle,
\]

where \( n \) denotes the number of quartets which can be formed with the valence nucleons outside the closed core, denoted by \( |0\rangle \). \( |\Theta_q\rangle \) is a condensate of the quartet \( Q_q^+ \) defined by

\[
Q_q^+ = \sum_{\ell=0}^{J_{max}} a_{J_{\ell}}^+(q_{\ell})_{J_{\ell}}^0,
\]

where

\[
(q_{\ell})_{J_{\ell}}^0 = \sum_{i_1,j_1,i_2,j_2} q_{i_1,j_1,i_2,j_2,J,T}^0 \left[(a_{i_1} a_{j_1})^T J_{T=0}^T [a_{i_2} a_{j_2}]^T J_{T=0}^T \right]
\]

In order to fix \( Q_q^+ \), we minimize the energy of the state \( |\Theta_q\rangle \) with respect to the coefficients \( q_{i_1,j_1,i_2,j_2,J,T}^0 \) and \( a_{i_1,j_1} \). The state (2) will be referred to as “ground” intrinsic state.

In addition to the state (2), we introduce a set of intrinsic states which are generated by promoting one of the quartets \( Q^+_q \) of \( |\Theta_q\rangle \) to an excited \( T = 0 \) configuration. These states have the general form [23]

\[
|\Theta_q\rangle = N_q Q_q^+ (Q^+_q)^{(n-1)}|0\rangle,
\]

with

\[
Q_q^{(k)} = \sum_{\ell=0}^{J_{max}} a_{J_{\ell}}^{(k)} (q_{\ell})_{J_{\ell}}^k,
\]

\[
(q_{\ell})_{J_{\ell}}^k = \sum_{i_1,j_1,i_2,j_2} q_{i_1,j_1,i_2,j_2,J,T}^{(k)} \left[(a_{i_1} a_{j_1})^T J_{T=0}^T [a_{i_2} a_{j_2}]^T J_{T=0}^T \right]
\]

Assuming that the quartet \( Q^+_q \) has already been fixed, we construct the new quartet \( Q^{(k)}_q \) by minimizing the energy of \( |\Theta_q\rangle \) with respect to the coefficients \( q_{i_1,j_1,i_2,j_2,J,T}^{(k)} \) and \( a_{i_1,j_1}^{(k)} \) (under the constraint of orthogonality when various states with the same \( k \) are involved). The states (5) will be identified with the value of the quantum number \( k \). In particular, for \( k = 0 \) and \( 2 \) (the states (5) will be referred to as “\( \beta \)" and “\( \gamma \)” intrinsic states, respectively. We will also indicate with the symbols \( q^+_J \) and \( q^+_J \) the associated quartets (7).

Having defined the quartets, we are ready to carry out configuration-interaction calculations in a space spanned by these quartets. To this purpose we define the set of states (we work in the \( m \)-scheme)

\[
|\Psi_{\{ J \}, M}\rangle = \prod_{J=0,1,\ldots, M} (q^+_J)^{N^{JM}}|0\rangle
\]

with the conditions

\[
\sum_{J=0}^{M} N^{JM} = n, \quad \sum_{JM} M N^{JM} = \bar{M}.
\]

We then orthonormalize the states (8) and diagonalize the Hamiltonian in this new basis for the various \( \bar{M} \). Calculations have been carried out in the \( sd \) and \( pf \) shells by adopting the USDB [24] and KB3G [25] interactions, respectively. The vacuum state \( |0\rangle \) in Eq. (8) stands for the nucleus \( \alpha \) for \( sd \) shell nuclei and for \( \alpha \) for \( pf \) shell nuclei.

In Figs. 1, 2 and 3, columns (A) and (B), we show what changes occur in the low-lying spectra of the nuclei under study when a set of dynamical quartets replace the analogous set of static quartets. These static quartets are those describing the lowest states of \( \alpha \) for the \( sd \) shell nuclei and those of \( \alpha \) for \( pf \) for \( \alpha \) for \( pf \) shell nuclei.

In the columns (A) we plot all the positive parity states up to the \( 61 \) obtained using a set of static \( J = 0,2,4 \) (plus \( J = 6 \) in the case of \( \alpha \) for \( pf \) quartets. In spite of the quite good results for the ground state correlation energy (defined as the difference between the ground state energies with and in absence of interaction), in all cases the spectra show marked differences with respect to the shell model results. In the columns (B) of the same figures, the dynamical quartets with the same \( J = 6 \) as above have been employed. In all three nuclei one observes a lowering of the yrast states \( J = 0,2,4,6 \) forming the ground state bands while most of the remaining states are pushed up in energy. The new ground state bands are all closer in energy to the shell model ones and a considerable improvement is observed also in the accuracy of the ground state correlation energies. Thus adopting the quartets \( q^+_J \) associated with \( |\Theta_q\rangle \) has had a positive effect only on the ground state bands of the nuclei under study.

Still in Figs. 1, 2 and 3 we observe how the use of the intrinsic states \( |\Theta_q\rangle \) (5) can help to improve the spectra of the nuclei under study. In \( \alpha \) for \( \alpha \) in Fig. 1(B), one notices that a band formed by the \( J = 2,3,4,5 \) states has been shifted higher in energy when passing from static to dynamical quartets. By making use of the definition of the \( \gamma \) intrinsic state, we construct new quartets \( q^{(k)}_J \) with \( J = 2,3,4 \) and perform a configuration-interaction calculation that includes these new quartets in addition to those already used for the calculation of Fig. 1(B). The new result is shown in Fig. 1(C). We observe a clear lowering of the energies of the states \( J = 2,3,4,5 \) while the states of the ground state band have remained basically unmodified with respect to those of column (B). The inclusion of the quartets derived with the help of the \( \gamma \) intrinsic state...
Figure 1. Spectra of $^{24}$Mg obtained by performing configuration-interaction calculations in spaces built with various sets of $T = 0$ quartets (see text): (A), $J = 0, 2, 4$ static quartets from $^{20}$Ne; (B), $J = 0, 2, 4$ dynamical quartets from the ground intrinsic state (2); (C), the same set as in (B) plus $J = 2, 3, 4$ quartets from the $\gamma$ intrinsic state; (D), the same sets as in (C) plus $J = 0, 2, 4$ quartets from the $\beta$ intrinsic state. SM, shell model results; EXP, experimental spectrum. The numbers above the symbols (A)-(D) are the relative errors in the ground state correlation energy with respect to the shell model value.

Figure 2. Spectra of $^{28}$Si obtained by performing configuration-interaction calculations in spaces built with various sets of $T = 0$ quartets (see text): (A), $J = 0, 2, 4$ static quartets from $^{20}$Ne; (B), $J = 0, 2, 4$ dynamical quartets from the ground intrinsic state (2); (C), the same set as in (B) plus $J = 2, 3, 4$ quartets from the $\gamma$ intrinsic state; (D), the same sets as in (C) plus $J = 0, 2, 4$ quartets from the $\beta$ intrinsic state. SM, shell model results; EXP, experimental spectrum. The numbers above the symbols (A)-(D) are the relative errors in the ground state correlation energy with respect to the shell model value.

Figure 3. Spectra of $^{48}$Cr obtained by performing configuration-interaction calculations in spaces built with various sets of $T = 0$ quartets (see text): (A), $J = 0, 2, 4, 6$ static quartets from $^{44}$Ti; (B), $J = 0, 2, 4, 6$ dynamical quartets from the ground intrinsic state (2); (C), the same set as in (B) plus $J = 2, 3, 4$ quartets from the $\gamma$ intrinsic state; (D), the same sets as in (C) plus $J = 0, 2, 4$ quartets from the $\beta$ intrinsic state. SM, shell model results; EXP, experimental spectrum. The numbers above the symbols (A)-(D) are the relative errors in the ground state correlation energy with respect to the shell model value.

has therefore essentially affected only those states which can be associated to a $\gamma$ band of $^{24}$Mg (and, in addition, the state $2_3$). As a final step, we have explored the effect on this spectrum of the inclusion of a set of quartets $(q_{13}^0)^4$, with $J = 0, 2, 4$. The basic effect which can be observed in Fig. 1(D) is a lowering of the states $J = 0_2, 2_3$ which can be associated to a $\beta$ band of $^{24}$Mg. As a result of the new diagonalization, also the $5_1$ state is lowered in energy. The final spectrum shows a good agreement with the shell model one.

For what concerns $^{28}$Si, what is most striking in the spectrum of Fig. 2(B) is the absence of a state $0_2$ close to the state $4_1$ as observed in both the shell model and the experimental spectrum. By interpreting this state $0_1$ as the head of a $\beta$ band, as a next step, we enlarge the model space by also including the quartets $(q_{13}^0)^4$, with $J = 0, 2, 4$ associated with the $\beta$ intrinsic state. The result of the new configuration calculation can be seen in Fig. 2(C). The inclusion of the new quartets mostly affects the yrare $J = 0, 2$ states by giving rise, in particular, to a surprising lowering of the $0_2$ state which positions itself immediately above the $4_1$ state, where it is expected to be. In spite of the fact that a reasonably good agreement with the shell model spectrum has already been achieved, we perform an additional calculation which also includes the quartets $(q_{13}^0)^4$, with $J = 3, 4$, associated with the $k = 3$ intrinsic state (5). Such a new calculation is stimulated by an old analysis of $^{28}$Si [26] in which the band head of a $K = 3$ band is positioned immediately above the $0_2$ state. As it can be seen in Fig. 2(D), the new calculation essentially lowers the energy of the $3_1$ and $4_2$ states and, in addition, that of the $3_2$ state. This new calculation further improves the quality of the QM spectrum which compares well with the shell model one.

The last case under investigation, $^{48}$Cr, shares some analogies with the corresponding one of $^{24}$Mg. Indeed one observes in Fig. 3(B) that, also in this case, the states $2_2, 3_1, 4_2$ have been shifted higher in energy when replacing the static quartets with the corresponding dynamical ones associated with the ground intrinsic state. These
states reminding those of a $\gamma$ band, we proceed as for $^{24}$Mg by introducing the quartets ($q_j^\alpha$)$_{\beta}$, associated with the $\gamma$ intrinsic state, with $J = 2, 3, 4$. The new calculation, Fig. 3(C), leaves unaffected the states of the ground rotational band (as for $^{24}$Mg) while it lowers significantly the states $2_2, 3_1, 4_2$. The new calculation also leads to the appearance of new states ($2_2, 4_1, 0_2$) in the highest part of the spectrum. These states have a correspondence with the shell model ones but with a $0_2$ still too high in energy. By interpreting this state as a possible head of a $\beta$ band and wishing to lower its energy, we perform a final calculation, Fig. 3(D), leaves unafected the states of the ground rotational band (as for $^{24}$Mg) while it lowers significantly the states $2_2, 3_1, 4_2$. The new calculation also leads to the appearance of new states ($2_2, 4_1, 0_2$) in the highest part of the spectrum. These states have a correspondence with the shell model ones but with a $0_2$ still too high in energy. By interpreting this state as a possible head of a $\beta$ band and wishing to lower its energy, we perform a final calculation which includes also the quartets ($q_j^\alpha$)$_{\beta}$ with $J = 0, 2, 4$. This calculation leads to a significant lowering of the $0_2$ state (Fig. 3(D)) which improves the agreement between exact and approximate spectra.

3 Summary and conclusions

In this paper we have provided a description of deformed $N = Z$ nuclei in a formalism of $\alpha$-like quartets. Quartets have been constructed variationally by resorting to the use of proper intrinsic states. Spectra have been obtained by carrying out configuration-interaction calculations in spaces built with these quartets. For each nucleus more sets of quartets have been used in correspondence with the various types of intrinsic states introduced. The intriguing aspect of these calculations has been the observation of band-like structures associated with the various sets of quartets. Also intriguing has been observing that these structures interfere little with each other in the sense that once one structure has been generated by means of a set of quartets, enlarging the configuration space with a new set of quartets moves down the states associated with the new structure without, however, strongly affecting the previous structure. The procedure has been applied to $^{24}$Mg, $^{28}$Si and $^{48}$Cr nuclei and it has provided a good description of the low-lying spectra in all three cases. A merit of this description is that of relying on only a few degrees of freedom. For what concerns the ground states, in particular, we have shown that already the $T = 0$ quartets with $J = 0$ and $J = 2$ can guarantee an accurate approximation of these states. As a general conclusion, the results achieved in this work promote the new method proposed for the definition of the quartets as well as the use of the latter as basic structures for an effective description of the ground and excited states of deformed $N = Z$ nuclei.

Acknowledgements

This work was supported by a grant of the Romanian Ministry of Research and Innovation, CNCS - UEFISCDI, project number PCE 160/2021, within PNCDI III.

References