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# Near-threshold hadron scattering with effective field theory

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**Abstract.** When an exotic hadron locates near the threshold with the channel couplings, the internal structure of the exotic hadron is related to the scattering length. To incorporate the threshold effect, the Flatté amplitude has been often used to determine the scattering length. It is however known that an additional constraint is imposed on the Flatte amplitude near the threshold. We discuss this problem by using the effective field theory for the coupled-channel scattering.

## 1 Introduction

Exotic hadrons, such as $T_{cc}$, $X(3872)$, and $f_0(980)$, are currently attracting attention [1, 2]. Many exotic hadrons are known to appear near the threshold of two hadron scattering. In such cases, the internal structure of exotic hadrons is strongly related to the scattering length. When there is not only one scattering channel but also decay channels, it is also necessary to consider the effect of the channel couplings. For the analysis of near-threshold exotic hadrons, the Flatté amplitude [3] is now often used, which includes the threshold effect. Since each component of the Flatté amplitude can be written in the form of the effective range expansion, the scattering length $a_{\rm F}$ can be determined from the Flatté amplitude.

However, the Flatté amplitude has the following problem; in the case of the two-channel scattering, the Flatté amplitude has three parameters, but the number of parameters is reduced to two near the threshold [4]. Thus, the Flatté amplitude is not a general amplitude in the threshold region, and the Flatté scattering length $a_{\rm F}$ may not be general. In more general frameworks, how would the scattering length be described?

## 2 Comparison of Flatté and EFT

We take the case of the two-channel scattering as an example and compare the Flatté amplitude with the general form of the scattering amplitude determined from the optical theorem. We consider the case that the threshold of channel 2 is higher than that of channel 1. The Flatté amplitude at energy $E$ is written by

$$f^{\rm F} = \frac{1}{-2E + 2E_{\rm BW} - ig_1^2 p - ig_2^2 k}\begin{pmatrix} g_1^2 & g_1 g_2 \\ g_1 g_2 & g_2^2 \end{pmatrix}, \tag{1}$$

where $g_1$ and $g_2$ represent the coupling contants and $E_{\rm BW}$ is the bare energy. $p(E)$ and $k(E)$ denote the momenta in channels 1 and 2, respectively. It is known that the Flatté amplitude

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satisfies the optical theorem. However, when $f^{\mathrm{F}}$ is expanded up to the first order in $k$ near the channel 2 threshold, it is known that the amplitude depends only on $R = g_2^2/g_1^2$ and $\alpha = 2E_{\mathrm{BW}}/(g_1^2 p_0)$, reducing the number of independent parameters to two [4], where $p_0 = p(E = 0)$ is the momentum of channel 1 at the threshold of channel 2.

On the other hand, K-matrix, M-matrix, etc. are known as general scattering amplitudes satisfying the optical theorem [5]. In this study, we use the EFT amplitude [6] derived from the non-relativistic effective field theory (EFT) with contact interactions:

$$f^{\mathrm{EFT}} = \frac{1}{\frac{1}{a_{12}^2} - \left(\frac{1}{a_{11}} + ip\right)\left(\frac{1}{a_{22}} + ik\right)} \begin{pmatrix} \frac{1}{a_{22}} + ik & \frac{1}{a_{12}} \\ \frac{1}{a_{12}} & \frac{1}{a_{11}} + ip \end{pmatrix}. \tag{2}$$

$a_{11}, a_{12}, a_{22}$ are the parameters of the EFT amplitude in units of the length. The EFT amplitude, similarly to the Flatté amplitude, contains three parameters and satisfies the optical theorem. We expand the EFT amplitude up to the first order in $k$ and it is shown that the number of parameters of the EFT amplitude remains three even near the threshold. Therefore, the use of the EFT amplitude solves the problem of the Flatté amplitude.

Although the EFT amplitude is found to be more general than the Flatté amplitude, the relationship between the EFT amplitude and the Flatté amplitude is not clear. This is because the EFT amplitude has an inverse matrix, while the Flatté amplitude does not, and thus the EFT amplitude and the Flatté amplitude cannot be directly mapped to each other. In order to clarify the relationship between the two, we construct a scattering amplitude that can represent both the EFT amplitude and the Flatté amplitude.

## 3 General amplitude

We introduce the general amplitude $f^{\mathrm{G}}$ with a new parametrization based on the EFT amplitude. $f^{\mathrm{G}}$ is represented by dimensionless constants $\gamma$ and $\epsilon$ and a parameter $A_{22}$ in units of the length:

$$f^{\mathrm{G}} = \frac{1}{-\frac{1}{A_{22}^2} - i\frac{1}{A_{22}}\epsilon p - i\frac{1}{A_{22}}k - \gamma pk} \begin{pmatrix} \frac{1}{A_{22}}\epsilon + i\gamma k & \frac{1}{A_{22}}\sqrt{\epsilon - \gamma} \\ \frac{1}{A_{22}}\sqrt{\epsilon - \gamma} & \frac{1}{A_{22}} + i\gamma p \end{pmatrix}. \tag{3}$$

When $\gamma = \epsilon$, the channel couplings vanish and $f^{\mathrm{G}}$ is written by $f^{\mathrm{G}} = (1/(-1/(A_{22}\epsilon) - ip), 1/(-1/(A_{22}) - ik))$. From this, $A_{22}$ represents the scattering length of channel 2 in the absence of the channel couplings.

Next, we discuss the relation between the general amplitude, the EFT amplitude, and the Flatté amplitude. When $\gamma = 0$, $f^{\mathrm{G}}$ is given as follows:

$$f^{\mathrm{G}} = \frac{1}{-\frac{1}{A_{22}} - i\epsilon p_0 - ik} \begin{pmatrix} \epsilon & \sqrt{\epsilon} \\ \sqrt{\epsilon} & 1 \end{pmatrix}. \tag{4}$$

This amplitude is equivalent to the Flatté amplitude up to first order in $k$. In other words, the general amplitude with $\gamma = 0$ reproduces the Flatté amplitude. Furthermore, the decreas of the Flatté amplitude parameters near the threshold can be understood from the condition $\gamma = 0$ in the general amplitude. In this case, the determinant of the general amplitude behaves as $\lim_{\gamma\to 0} \det\left(f^{\mathrm{G}}\right) = 0$. From this feature, the inverse of $f^{\mathrm{G}}$ does not exist when $\gamma = 0$. Since general amplitude is obtained by a different parametrization of the EFT amplitude, for $\gamma \neq 0$, the general amplitude corresponds to the EFT amplitude. In other words, both the Flatté amplitude and the EFT amplitude can be obtained from the general amplitude by choosing the parameter $\gamma$.

Next, we perform the effective range expansion for $f^{\mathrm{G}}$ in terms of $k$, and determine the scattering length. We expand the denominator of $f^{\mathrm{G}}_{22}$ in powers of the momentum $k$:

$$f^{\mathrm{G}}_{22} = \frac{1}{-\frac{1}{A_{22}}\left(\frac{\frac{1}{A_{22}}+i\epsilon p_0}{\frac{1}{A_{22}}+i\gamma p_0}\right) - \frac{i(\epsilon-\gamma)}{2(1+iA_{22}\gamma p_0)p_0^2}k^2 - ik + O(k^4)}. \tag{5}$$

This shows that, $f^{\mathrm{G}}_{22}$ can be written as the effective range expansion in $k$, and we can define the scatterng length $a_{\mathrm{G}}$ in the general amplitude as follows:

$$a_{\mathrm{G}} = A_{22}\left(\frac{\frac{1}{A_{22}} + i\gamma p_0}{\frac{1}{A_{22}} + i\epsilon p_0}\right). \tag{6}$$

In the same way, we expand $f^{\mathrm{G}}_{11}$ in $k$:

$$f^{\mathrm{G}}_{11} = \frac{\frac{\epsilon^2}{\epsilon-\gamma}}{-\frac{1}{A_{22}}\frac{\epsilon}{\epsilon-\gamma} - i\frac{\epsilon^2}{\epsilon-\gamma}p_0 - \left(A_{22}\frac{\gamma}{\epsilon} + i\frac{\epsilon^2}{2(\epsilon-\gamma)p_0}\right) - ik + O(k^3)}. \tag{7}$$

Because the power series in Eq. (7) contains terms such as $k^3$, $f^{\mathrm{G}}_{11}$ cannot be written in the form of the effective range expansion. Also, the coefficients of each term in the denominator of $f^{\mathrm{G}}_{11}$ are different from those of $f^{\mathrm{G}}_{22}$ in Eq. (5). In particular, the constant term in the denominator of $f^{\mathrm{G}}_{11}$ is different from the scattering length in Eq. (6). On the other hand, as can be seen from Eq. (1), the coefficients of the power series of the Flatté amplitude are common for all the components, and the constant terms of the denominator of the scattering amplitudes are entirely given by the Flatté scattering length.

To summarize, from Eqs. (5) and (7), in general, the $f_{22}$ component can be written as the effective range expansion near the threshold of channel 2, but the $f_{11}$ component cannot be written and the scattering length cannot be defined. On the other hand, when $\gamma = 0$, $f^{\mathrm{G}}$ reduces to the Flatté amplitude, and the scattering length $a_{\mathrm{G}}$ becomes the Flatté scattering length $a_{\mathrm{F}}$ as follows:

$$a_{\mathrm{G}} = A_{22}\left(\frac{\frac{1}{A_{22}} + i\gamma p_0}{\frac{1}{A_{22}} + i\epsilon p_0}\right) \xrightarrow{\gamma=0} \frac{1}{\frac{1}{A_{22}} + i\epsilon p_0} = a_{\mathrm{F}}. \tag{8}$$

Similarly, the constant term of the denominator of $f^{\mathrm{G}}_{11}$ in Eq. (7) becomes $a_{\mathrm{F}}$:

$$\frac{1}{\frac{1}{A_{22}}\frac{\epsilon}{\epsilon-\gamma} + i\frac{\epsilon^2}{\epsilon-\gamma}p_0} \xrightarrow{\gamma=0} \frac{1}{\frac{1}{A_{22}} + i\epsilon p_0} = a_{\mathrm{F}}, \tag{9}$$

In general, if $\gamma$ is nonzero, the constant term in the denominator of $f^{\mathrm{G}}_{11}$ is different from the correct scattering length $a_{\mathrm{G}}$, so an analysis using the Flatté amplitude where the scattering length appears in $f_{11}$ may not give the correct scattering length.

## 4 Application

In order to verify the effect of the value of $\gamma$ on the scattering length $a_{\mathrm{G}}$, we fix the constant term of the denominator of $f^{\mathrm{G}}_{11}$ and vary $\gamma$. In this study, we consider the $\pi\pi$-$K\bar{K}$ system with $f_0(980)$, which has already been analyzed by the Flatté amplitude. In Ref. [7], the constant term in the denominator of $f_{\pi\pi}$ corresponding to $f^{\mathrm{G}}_{11}$ is determined to be $-1.0-1.0i$ GeV in the

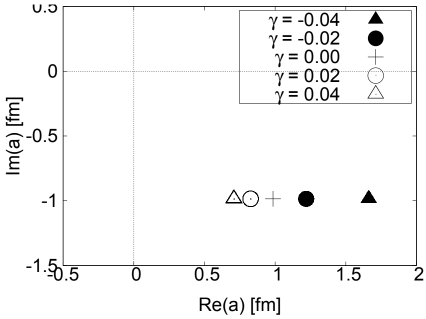


**Figure 1.** Real and imaginary parts of the scattering length when $\gamma$ is varied from $-0.04$ to $+0.04$. The cross represents the scattering length of the Flatté amplitude with $\gamma = 0$.

analysis using the Flatté amplitude. In this case, two conditions are imposed to the parameters $A_{22}, \gamma, \epsilon$. In order to compare the scattering lengths $a_{\mathrm{G}}$ and $a_{\mathrm{F}}$ of the Flatté amplitude, we calculate the scattering length $a_{\mathrm{G}}$ with difference values of $\gamma$. The change of the scattering length $a_{\mathrm{G}}$ when $\gamma$ is varied from $-0.04$ to $+0.04$ is shown in Fig.1.

In Fig. 1, the point represented by the cross ($\gamma = 0$) corresponds to the scattering length $a_{\mathrm{F}}$ of the Flatté amplitude, and the general scattering length $a_{\mathrm{G}}$ deviates from $a_{\mathrm{F}}$ by $\sim 0.5$ fm when $\gamma$ is changed from $-0.04$ to $+0.04$. In the present case, the imaginary part of $a_{\mathrm{G}}$ does not depend on $\gamma$ as seen in Fig. 1. This property can be analytically shown by the imaginary part of Eq. (6). We find that the scattering length $a_{\mathrm{G}}$ varies quantitatively for different $\gamma$. Therefore, the scattering length determined from the Flatté amplitude $a_{\mathrm{F}}$ with $\gamma = 0$ may deviate from the correct scattering length $a_{\mathrm{G}}$ with $\gamma \neq 0$ numerically.

## 5 Summary

In this study, we discuss the properties of general scattering amplitudes with the channel couplings. First, the EFT amplitude and the Flatté amplitude are compared, showing that the EFT amplitude does not reduce to the Flatté amplitude directly. Next, we solve this problem by introducing a new parametrization of the EFT amplitude to construct the general amplitude that includes both the EFT amplitude and the Flatté amplitude. Finally, by applying the general amplitude to the $\pi\pi$-$K\bar{K}$ system and quantitatively comparing the correct scattering length with the one determined from the Flatté amplitude, we show that the scattering length of the Flatté amplitude may deviate from the correct scattering length by about 0.5 fm.

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