Far-off-equilibrium early-stage dynamics in high-energy nuclear collisions

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Abstract. We explore the far-off-equilibrium aspects of the (1+1)-dimensional early-stage evolution of a weakly-coupled quark-gluon plasma using kinetic theory and hydrodynamics. For a large set of far-off-equilibrium initial conditions the system exhibits a peculiar phenomenon where its total equilibrium entropy decreases with time. Using a non-equilibrium definition of entropy based on Boltzmann’s H-function, we demonstrate how this apparently anomalous behavior is consistent with the second law of thermodynamics. We also use the H-function to formulate ‘maximum-entropy’ hydrodynamics, a far-off-equilibrium macroscopic theory that can describe both free-streaming and near-equilibrium regimes of quark-gluon plasma in a single framework.

1 Introduction

Precise determination of transport coefficients like the specific shear and bulk viscosities, \( \eta / s \) and \( \zeta / s \), of the quark-gluon plasma formed in high-energy nucleus-nucleus collisions hinges upon accurately modeling the stress tensor (\( T^{\mu \nu} \)) evolution during the system’s early stage. This stage is characterised by far-off-equilibrium dynamics which may be modeled by weakly coupled kinetic theory until \( O(1) \) fm/c [1, 2]. This approach is, however, numerically daunting as solving kinetic theory amounts to tackling a 7-dimensional problem in phase-space. Moreover, if one is only interested in the evolution of macroscopic quantities like \( T^{\mu \nu} \), solving for the full kinetic distribution is likely unnecessary. It is thus desirable to have a macroscopic framework which can model the far-off-equilibrium evolution of \( T^{\mu \nu} \) both physically accurately and numerically efficiently. In this work, we first explore the sensitivity of the \( T^{\mu \nu} \) evolution in kinetic theory to initial state momentum anisotropies of the plasma. By considering extreme off-equilibrium initial conditions for a quark-gluon gas undergoing Bjorken expansion [3], we point out non-intuitive out-of-equilibrium effects arising in kinetic theory. In the second part we formulate a new macroscopic theory (ME-hydrodynamics) which can be used to describe in a single framework both the far-off-equilibrium pre-hydrodynamic and the near-equilibrium dissipative hydrodynamic regimes of the plasma.

2 Kinetic theory of a massive quark-gluon gas

For a weakly interacting gas of quarks, anti-quarks, and gluons undergoing boost-invariant Bjorken expansion along the beam axis, we solve the Boltzmann equation in a relaxation-time
approximation,

$$\frac{\partial f^i}{\partial \tau} = -\frac{1}{\tau_R(T)} \left( f^i - f^i_{\text{eq}} \right).$$  \hspace{1cm} (1)

Here \( \tau \) is Milne time, \( \tau_R \) is the microscopic relaxation time, and the superscript \( i \in \{ q, \bar{q}, g \} \) on the kinetic distributions distinguishes between particle species. \( f^i_{\text{eq}} \) are given by Fermi-Dirac (for quarks and anti-quarks) or Bose-Einstein (for gluons) distributions which involve the Landau matched effective temperature and quark chemical potential \((T, \mu)\). Symmetries of Bjorken flow imply vanishing net-quark diffusion, i.e. \( n(\tau) \propto 1/\tau \) and \( T_{\mu\nu} = \sum_i p^i_\mu p^i_\nu f^i = \text{diag}(e, P_T, P_T, P_L) \), where \( e \) is energy density and \( P_T \) and \( P_L \) are effective transverse and longitudinal pressures. An important physical quantity is the non-equilibrium entropy density (in the rest frame of a fluid having velocity \( u^\mu \)), obtained from Boltzmann’s H-function:

$$s = -\sum_i \int_{p_i} \left( u^i \cdot p_i \right) \left( f^i \ln f^i - \frac{1 + a_i f^i}{a_i} \ln \left( 1 + a_i f^i \right) \right),$$  \hspace{1cm} (2)

where \( a_{q, \bar{q}} = -1 \) and \( a_g = 1 \). In equilibrium \( s \rightarrow s_{\text{eq}} = (e + P - \mu n)/T \). In Fig. 1 we show solutions of kinetic theory for two sets of extreme far-off-equilibrium initial conditions (see figure caption) which were set up using a Romatschke-Strickland (RS) distribution [5, 6]. Although all curves start with the same effective \((T, \mu_B)\), the phase trajectories are quite sensitive to the choice of initial momentum space anisotropy. In Bjorken flow, Navier-Stokes hydrodynamics predicts that the ratio \( s_{\text{eq}}/n \) must increase over time due to viscous heating. While this is indeed the case for panel (a) (see dotted lines for \( s_{\text{eq}}/n \) evolution in (b)), this expectation is not borne out for the trajectories in panel (c). Here, \( s_{\text{eq}}/n \) decreases for a certain duration of time. However, this does not imply a violation of the second-law of thermodynamics as the total entropy per baryon which includes non-equilibrium effects never decreases. The feature of decreasing equilibrium entropy per baryon density results in a peculiar phenomena which we call ‘non-equilibrium cooling’ (see Fig. 2). Here, the effective temperature falls even faster than what is expected for an ideal (inviscid) fluid.

### 3 Maximum-entropy truncation of the Boltzmann equation

The Boltzmann equation can be expressed as an infinite hierarchy of equations for momentum moments of \( f(x, p) \) [7] where low-order moments corresponding to components of \( T^{\mu\nu} \)
are coupled to higher-order ‘non-hydrodynamic’ moments. To obtain a macroscopic
description solely in terms of \( T^{\mu \nu} \), the infinite hierarchy has to be truncated by expressing the
non-hydrodynamic moments in terms of an approximate kinetic distribution using only inform-
ation contained in \( T^{\mu \nu} \). Based on Jaynes’s insights on the connections between statistical
mechanics and information theory [8], Everett et al. [9] recently proposed a novel way of
reconstructing a kinetic distribution from the energy-momentum tensor using the maximum
entropy principle. The idea is to find an \( f(x, p) \) that maximizes the non-equilibrium entropy
density (2), subject to the information (constraint) that it reproduces the given 10 components
of \( T^{\mu \nu} \). For a single component gas the maximum entropy distribution is [9]

\[
f_{\text{ME}}(x, p) = \left[ \exp \left( \frac{\Lambda_{\mu \nu} p^\mu p^\nu}{u \cdot p} \right) - a \right]^{-1},
\]

where \( \Lambda_{\mu \nu} \) are Lagrange multipliers corresponding to \( T^{\mu \nu} \). Landau matching conditions fur-
ther simplify the argument of the exponential [10]. Unlike the commonly used distributions
for Grad or Chapman-Enskog (CE) truncation, \( f_{\text{ME}} \) is positive definite for all momenta and al-
low for non-equilibrium matching to conserved currents for a wide range of non-equilibrium
stresses. It also ensures that the resulting macroscopic framework, which we call ME-hydro,
has a non-negative entropy production rate [11] and that in the limit of small viscous stresses
ME-hydro reduces to second-order Chapman-Enskog fluid dynamics [9].

4 ME-hydro vs. RTA kinetic theory in Bjorken and Gubser flows
The exact evolution equations for the 3 independent components of \( T^{\mu \nu} = \text{diag}(e, P_T, P_L) \)
in Bjorken flow are given by

\[
\begin{align*}
\frac{d e}{d \tau} &= -\frac{e + P_L}{\tau}, \\
\frac{d P_T}{d \tau} &= -\frac{P_T - P}{\tau_R} - \frac{P_T}{\tau} + \frac{\zeta_T}{\tau}, \\
\frac{d P_L}{d \tau} &= -\frac{P_L - P}{\tau_R} - \frac{3P_L}{\tau} + \frac{\zeta_L}{\tau}.
\end{align*}
\]

The terms \( \zeta_T, \zeta_L \) introduce couplings to ‘non-hydrodynamic’ moments of \( f(\tau, p_T, p_z) \); for
example, \( \zeta_L = \int_p E^2 p^4 f \). To truncate we replace \( f \mapsto f_{\text{ME}} \) where \( f_{\text{ME}} \) is constructed using
evolution equations

for the two independent (dimensionless) variables \( (\hat{e}, \hat{P}_T) \) as functions of de-Sitter ‘time’ \( \rho \) are
equilibrium. Figure 4a shows that Chapman-Enskog hydrodynamics [14] fail to capture the late-time transverse free-streaming regime of Gubser flow. The only framework that performs slightly better than ME-hydro is anisotropic hydrodynamics [15, 16] (shown in panel (b)) which uses the RS ansatz as a truncation distribution.

4.1 Summary
Non-equilibrium effects during the early stages of QGP evolution can substantially alter its phase trajectories as compared to near-equilibrium predictions. ME-hydrodynamics, a macroscopic theory based on a simple physical principle, holds promise in describing such far-off-equilibrium effects. Further numerical analysis is required to test this expectation.

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References