Theoretical applications to nuclear astrophysics

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Abstract. Nuclear reaction study is important to understand elemental abundance in the universe and stability of stars. For several reactions due to the paucity of experimental data or difficulties in performing the direct measurements, we depend upon theoretical estimates and therefore one needs reliable theoretical models with minimum inputs. This paper is a short review of our work on nuclear reactions and their applications to astrophysics. In particular, we discuss the nuclear transfer and breakup reaction while focusing on our theoretical modelings and techniques involved.

1 Introduction

Nuclear reactions play an important role in the formation of elements in the universe, stability of stars and in the study of nuclear structure as well. It is now well established that the elements which we see on earth and in our body, have been produced billions of years ago via different kinds of nuclear reactions that occurred after the Big Bang and in the stellar environment. Therefore, the information of cross sections and rates of these reactions is necessary in order to understand the origin of elements in the Universe. As energies corresponding to the stellar temperature lie from few tens to hundreds of keV/u, the direct measurements in the lab become often difficult because of the small cross sections, low yields and due to electron screening effects at such low energies. In several such cases, we therefore, depend upon theoretical estimates of reaction cross section and subsequent rates. Alternatively, indirect methods such as the Asymptotic Normalization Coefficient method, the Coulomb dissociation method, the Trojan horse method and (d, p) reactions are also being used widely [1, 2]. These methods are again a combination of theory as well as measurements.

Therefore, reliable reaction theories are always desired in this regard. Over the years, various theoretical models have been developed for different types of nuclear reactions within different frameworks [like distorted wave Born approximation, coupled channel method, adiabatic method, Ab Initio theories etc. (see for example Refs. [3–5])] and many of those work quite well but of course, have their own limits of validity. Similarly, other frameworks like potential models and statistical models are also popular [6, 7]. Phenomenological $R$-matrix method [8] which was initially developed to parametrize resonances is also quite successful in predicting cross sections and widths etc. [9–11].

In this paper, we discuss the nuclear transfer reactions and Coulomb breakup reactions with their applications to nuclear astrophysics where they are used indirectly to extract the important information for low energy astrophysical reactions. Mainly, we focus on the full quan-
Coulomb breakup is one of the methods used to study the radiative capture reactions. At astrophysical energies, when direct measurements of the reaction $b + c \rightarrow a + \gamma$ in the lab become difficult, one can extract the desired capture cross section from the time reversal process i.e the photodisintegration. The latter can be related with the Coulomb breakup of a over some heavy target measured at higher energies, where the problem of small cross sections can be avoided, see for example Refs. [1, 2]. Experimentally, the method is mainly applicable when the capture leads to the formation of the ground state. In the literature there exist different theoretical models of Coulomb breakup, starting from the semi-classical theory [24] to fully quantum mechanical theory [12] and Coupled channel approach as well [3]. For more details about these models one is referred to Ref. [4] and references therein. Here we constrain ourself to the Coulomb breakup within the post-form FRDWBA formalism only.

If we consider the elastic breakup of projectile $a$ into fragments $b$ and $c$ in the Coulomb field of some heavy target $t$, where $c$ is a neutron (see Ref.[25], for the case of three-charged particles in final state), then the relative energy spectra for the process can be written as

$$\frac{d\sigma}{dE_{\text{rel}}} = \frac{\mu_{bc} \mu_{at} P_{bc} P_{at}}{(2\pi)^5 \hbar^2 v_{at}} \int \sum_{l,m} |\beta_{lm}|^2 \frac{1}{2l + 1} d\Omega_{bc} d\Omega_{at}, \quad (1)$$

where $E_{\text{rel}}$ is the relative energy of fragment $b$ and $c$, $v_{at}$ is the relative velocity of $a - t$ in the entrance channel. $\Omega$, $\mu$ and $p$ represent the solid angles, reduced mass and linear momenta of the concerned systems, respectively. $l$ and $m$ are the relative orbital angular momentum and its projection, respectively. $\beta_{lm}$ is the reduced transition amplitude, which is given by [12]

$$\beta_{lm}(q_b, q_c; q_a) = \left\langle \chi_b^{(-)}(q_b, r) \chi_c^{(-)}(q_c, r) \right| V_{bc}(r_1) \left| \phi_a^{lm}(r_1) \chi_a^{(+)}(q_a, r) \right\rangle, \quad (2)$$

where $\chi_j^{(\pm)}$ are the pure Coulomb distorted wave functions of the concerned particles with incoming or outgoing boundary conditions and for the case of neutron it is a plane wave. $\phi_a^{lm}$ is the ground state wave function of the projectile, $V_{bc}$ is the bound state potential of $b - c$ system, whereas $r_j$ and $q_j$ are the Jacobi co-ordinates and wave vectors of the concerned particle (see Ref. [12] for more details).

Once the relative energy spectrum [Eq. (1)], is calculated, then one can relate it with the photodisintegration cross section $(\sigma_{\gamma \pi}^{\pi \lambda})$ using the relation [26]

$$\frac{d\sigma}{dE_{\text{rel}}} = \frac{1}{E_{\gamma}} \sum_{\pi, \lambda} \sigma_{\gamma \pi}^{\pi \lambda} n_{\pi \lambda}, \quad (3)$$

where $n_{\pi \lambda}$ is the equivalent photon number with multipolarity $\lambda$ and $\pi$ stands for either electric or magnetic transition. $E_{\gamma}$ is the photon energy that is related to the Q-value by the relation $E_{\gamma} = E_{\text{rel}} + Q$. If a single multipolarity dominates, then using the above relation one can estimate the photodisintegration cross section. From the photodisintegration cross sections one can obtain the desired capture cross section using the principle of detailed balance.

The theory has been used to estimate the capture cross section and subsequent reaction rates for several neutron capture reactions like $^8\text{Li}(n, \gamma)^9\text{Li}, ^{14}\text{C}(n, \gamma)^{15}\text{C}, ^{15}\text{N}(n, \gamma)^{16}\text{N},$
Nuclear transfer reactions provide a useful tool to study the structure of a nucleus [27] and to extract the important information on bound states and on low energy resonances crucial in the study of some astrophysical reactions when the latter are difficult to measure directly in the lab, see for example Ref. [28]. By comparing the measured angular distribution of the transfer cross sections with the calculated ones, we can find the angular momentum, spin-parity, spectroscopic factor (SF) or the asymptotic normalization coefficients (ANC) (defined as the amplitude of the tail of the overlap function) of the participating state of the residual nucleus. In case the final state is a resonance, one can estimate the width of the resonance by knowing the SF or the ANC [1].

There exist different frameworks of transfer reactions like DWBA, adiabatic method, Continuum discretized coupled channel method (CDCC) and Faddeev’s method (see for example Refs. [3, 29]). DWBA being the simplest framework of transfer reactions is widely used to analyse the data. Here the transfer is considered as a one-step process and only the elastic channels are considered. Other frameworks, like CDCC and Faddeev’s method take into account the breakup channels too but are computationally expensive and demand efficient numerical techniques that can provide the faster and accurate computations. As a first step in this regard, in Refs. [21] we apply the Lagrange-mesh [30] and R-matrix methods [31] to transfer reactions in the DWBA framework. Both these techniques are known for providing a faster and accurate computations. Even for the simpler DWBA framework, in optimal conditions our calculations were about two times faster than the existing codes.

In the DWBA framework, the scattering matrix $U_{\alpha\beta}^{J\pi}$ ($J, \pi$ stands for the total spin and parity, respectively) for the process $A(= a + c) + t \rightarrow a + B(= t + c)$, where fragment c that can be a nucleon or a cluster, is transferred from the projectile $A$ to the target $t$ (spin-less in our case), is given by

$$U_{\alpha\beta}^{J\pi} = -i \frac{\sqrt{S_A S_B}}{\hbar} \int \chi_{L_A}^{J\pi}(R) K_{\alpha\beta}^{J\pi}(R, R') \chi_{L_B}^{J\pi}(R') RR' dR dR', \quad (4)$$

where $S_i$ is the spectroscopic factor of the concerned bound state of nucleus $i$. $\chi_{L_A}^{J\pi}(R)$ and $\chi_{L_B}^{J\pi}(R')$ are the radial scattering wave functions for the relative motion of $A - t$ and $a - B$ nuclei, respectively, with $L_A$ and $L_B$ being the orbital angular momenta in the entrance and exit channels. $R$ and $R'$ are the radial co-ordinates for the relative motion of $A$ with $t$ and $a$ with $B$ in the initial and final channel, respectively. For a schematic diagram of the transfer process see Ref. [23]. Labels $\alpha$ and $\beta$ stand for $(L_A, \ell_A, I_A)$ and $(L_B, \ell_B, I_B)$, respectively, where $L_j$ and $\ell_j$ represent the spin and orbital angular momentum of nucleus $j$. $K_{\alpha\beta}^{J\pi}(R, R')$ is the transfer kernel, which is given by

$$K_{\alpha\beta}^{J\pi}(R, R') = J\langle Y_{L_A}(\Omega) \otimes \Phi^{I_A}_{\ell_A}(r_A) \rangle J\langle Y_{L_B}(\Omega') \otimes \Phi^{I_B}_{\ell_B}(r_B) \rangle J, \quad (5)$$

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where Φ’s are the bound state wave functions of the respective nucleus. \(r_A\) and \(r_B\) are the distances between clusters \(a - c\) and \(c - t\) inside nucleus A and B, respectively. \(J\) is the Jacobian which is introduced for taking care of the co-ordinate transformations. \(V\) is the interaction that depends upon the choice of post or prior-form of the scattering matrix. It was shown in Ref. [23] that with the full interaction term (including the remnant terms), prior and post-form DWBA calculations are equivalent. For more details and to understand the importance of remnant terms, one is referred to Refs. [21, 23].

Scattering matrix (4) involves scattering wave functions \(\chi_k\) and bound state wave functions \(\Phi_k\), which we calculate using Lagrange-mesh and \(R\)-matrix methods. In the \(R\)-matrix method, we divide the configuration space into two parts – internal and external, at some channel radius \(R_0\). In the external region the wave function takes the asymptotic form in terms of Coulomb incoming and outgoing functions and can be written as

\[
\chi^L_{\text{ext}}(R) = \frac{1}{\sqrt{v}} (I_L(kR) - U_L O_L(kR)),
\]

with \(k\) and \(v\) as the wave number and velocity of the concerned particle, respectively. The elastic scattering matrix \(U_L\) can be calculated from the properties of the Hamiltonian in the internal region \((R \leq R_0)\) [31] where the wave function is expanded over a set of \(N\) basis functions \(\varphi_i(R)\). In our case we use the Lagrange functions whose choice depends on the interval considered [30]. For a finite interval we use a Legendre basis whereas for the infinite interval a Lagrange–Laguerre basis is the best choice. Therefore, in the internal region we have

\[
\chi^L_{\text{int}}(R) = \sum_{i=1}^{N} c_i^L \varphi_i(R).
\]

The expansion coefficients \(c_i\) can be obtained in the matrix form by making the Hamiltonian Hermitian in the internal region \([0, R_0]\) with the help of Bloch operator, which also ensures the continuity of the derivative of the wave function at the boundary. The continuity condition of the wave function then provides the elastic scattering matrix \(U_L\) in terms of the \(R\)-matrix, defined as the reciprocal of the logarithmic derivative of the wave function at \(R_0\). Both the scattering and bound state wave functions in the initial and final channels can be calculated in this way but with a different choice of the basis functions as mentioned earlier. With the application of these methods, the scattering matrix (4) takes a simpler form and for a sufficient large radius its external part can be neglected. For more details one is referred to Refs. [21–23, 30, 31].

We have shown in Refs. [21, 23] that typically \(N \approx 30 – 40\) basis functions are sufficient to achieve a good convergence which are of course significantly less than those required in finite-difference methods (of the order of 500). One should also note that in this version of the \(R\)-matrix, the channel radius is not a fitting parameter. In the standard \(R\)-matrix approach, results should be independent of its value if chosen large enough.

In Refs. [21–23], we applied the above formalism to study the \(^{16}\text{O}(d, p)^{17}\text{O}, \(^{16}\text{O}(d, n)^{17}\text{F}\) and \(^{12}\text{C}(\text{Li}, t)^{16}\text{O}\) reactions. We studied the sensitivity of cross sections to bound state wave functions, peripherality of the reaction and the importance of the remnant interactions in these cases. Indeed to extract the ANC from the transfer reaction, one needs to ensure that the reaction is a peripheral. In our study of the \(^{12}\text{C}(\text{Li}, t)^{16}\text{O}\) reaction, which has been used in several indirect measurements of the astrophysically important reaction \(^{12}\text{C}(\alpha, \gamma)^{16}\text{O}\), we found that the spectroscopic factor of the \(0^+\) (6.05 MeV) state of \(^{16}\text{O}\) increased by 30 – 50% whereas for the \(2^+\) (6.92 MeV) state it increased by 6 – 14% [23]. In Ref. [32], we

\(^{1}\text{Determines the C/O abundance ratio after the He burning stage and also decides the fate of the stars after death.}\)
extended the applications of the model to the reactions involving resonances. Calculations in the framework of adiabatic distorted wave approximation were presented in Ref. [33].

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References