Photon-photon scattering

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Abstract. The existence of photon-photon scattering historically was one of the first non-trivial predictions of QED. However, since the cross section is very small at low energies it was only in 2017 that a direct measurement was achieved in heavy-ion collisions at the LHC. After short reviews of the history of the subject and of the CERN experiment, I discuss the general structure of the four-photon tensor, which also serves as the prototype for all vertices of four gauge bosons. I then come to generalizations from QED to the Standard Model and beyond, in particular to the Born-Infeld theory that has gained some popularity in recent years as an alternative to standard quantum electrodynamics.

1. A short history of light-by-light scattering

Let us start with a short review of early quantum electrodynamics, for whose development light-by-light scattering was pivotal (see [1] for an excellent historical review of early QED).

In 1928 Dirac developed the Dirac equation and the relativistic quantum mechanics of the electron, in 1930 hole theory and the foundations of quantum electrodynamics, leading to the prediction of the positron. His theory contains negative-energy states, which initially seemed problematic, but eventually led to the concept of a vacuum being filled by virtual particles. As early as 1931, Sauter realized that this should lead to the creation of electron-positron pairs by "vacuum tunneling" in a sufficiently strong electric field [2]: if a virtual pair separates out far enough along the field lines to draw their rest mass energy from the field it can turn real. This process is often depicted as if the field was acting on the virtual particles as if they were already real, driving them apart (Fig. 1) [1] .

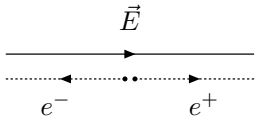


Figure 1. Vacuum pair creation by an external field.

[1] In reality the precise way of how real particles are formed in this pair-production process is very complex and still a matter of ongoing research (see, e.g., [3]).

Two years later the American physicist Halpern pointed out [4] that those virtual electron-positron pairs should make themselves felt also more indirectly by introducing an interaction between real photons (Fig. 2).

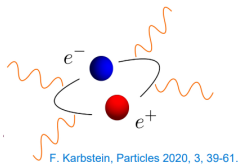
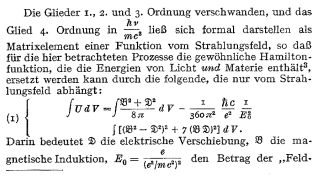


Figure 2. Effective interaction between real photons mediated by virtual electron-positron pairs.

Another American physicist, P. Debye, then conjectured that this effective photon-photon interaction is the reason for the solar corona. As quoted in [1], Debye then got W. Heisenberg interested in this effective interaction, who asked his students H. Euler and B. Kockel to calculate it for the simplest case, namely in the low-energy limit and to quartic order in the field. They were able to do this, and in 1935 published the result [5] in terms of a quartic effective Lagrangian written in terms of the two Maxwell invariants $\mathbf{B}^2 - \mathbf{D}^2$, $(\mathbf{B}\cdot\mathbf{D})^2$ (Fig. 3):



Die Glieder 1., 2. und 3. Ordnung verschwanden, und das Glied 4. Ordnung in $\frac{h\nu}{mc^2}$ ließ sich formal darstellen als Matrixelement einer Funktion vom Strahlungsfeld, so daß für die hier betrachteten Prozesse die gewöhnliche Hamiltonfunktion, die die Energien von Licht *und* Materie enthält[8], ersetzt werden kann durch die folgende, die nur vom Strahlungsfeld abhängt:

$$(1)\quad \left\{ \begin{aligned} &\int U\,dV = \int \frac{\mathfrak{B}^2 + \mathfrak{D}^2}{8\pi}\,dV - \frac{1}{360\pi^2}\,\frac{\hbar c}{e^2}\,\frac{1}{E_0^2} \\ &\int [(\mathfrak{B}^2 - \mathfrak{D}^2)^2 + 7\,(\mathfrak{B}\,\mathfrak{D})^2]\,dV. \end{aligned} \right.$$

Darin bedeutet $\mathfrak{D}$ die elektrische Verschiebung, $\mathfrak{B}$ die magnetische Induktion, $E_0 = \frac{e}{(e^2/mc^2)^2}$ den Betrag der „Feld-

Figure 3. Euler and Kockel's quartic effective Lagrangian (from [5]).

They also calculated the total cross section,

$$\sigma = \frac{973\ \alpha^4}{10125\ m^8}\frac{\omega^6}{\pi}\,. \tag{1}$$

Due to the factor $\sim \omega^6$ (which is, incidentally, a direct consequence of gauge invariance) this is exceedingly small for optical frequencies.

Shortly later, Heisenberg and Euler published their famous calculation of this effective Lagrangian to all orders in the field (" Euler-Heisenberg Lagrangian") [6],

$$\mathcal{L}(a,b) \quad = \quad -\frac{1}{8\pi^2}\int_0^\infty \frac{dT}{T^3}\,\mathrm{e}^{-m^2T}\left[\frac{(eaT)(ebT)}{\tanh(eaT)\tan(ebT)} - \frac{e^2}{3}(a^2-b^2)T^2 - 1\right]. \tag{2}$$

Here a, b are the two Maxwell invariants, related to $\mathbf{E}$, $\mathbf{B}$ by $a^2 - b^2 = B^2 - E^2$, $\quad ab = \mathbf{E}\cdot\mathbf{B}$. The leading quartic term is the one found by Euler and Kockel. However, this is still a low-energy approximation; the full effective Lagrangian has an infinite number of additional terms involving not only $F_{\mu\nu}$ but its derivatives to arbitrary orders.

If the field has an electric component ($b \neq 0$) there are poles on the integration contour at $ebT = k\pi$ which yield an imaginary part. This fact and its relation to Sauter's pair creation were noted by Heisenberg and Euler, but further investigated only in 1951 by Schwinger [7],

who obtained the following representation for the imaginary part of the purely electric Euler-Heisenberg Lagrangian,

$$\mathrm{Im}\mathcal{L}(E) \quad = \quad \frac{m^4}{8\pi^3}\beta^2 \sum_{k=1}^{\infty} \frac{1}{k^2} \exp\left[-\frac{\pi k}{\beta}\right] \tag{3}$$

($\beta = eE/m^2$). We note two important properties:

- The kth term relates through the optical theorem to the coherent creation of k pairs in one Compton volume.
- $\mathrm{Im}\mathcal{L}(E)$ depends on E nonperturbatively, which is a confirmation of the Sauter tunneling picture.

The Euler-Heisenberg Lagrangian can be directly used to deduce the properties of the QED vacuum as a non-linear optical medium (see, e.g., [8]). Moreover, it holds the full information on the one-loop N photon amplitudes in the low energy limit (where all photon energies are small compared to the electron mass, $\omega_i \ll m$). From the weak field expansion coefficients c_{kl}, defined by

$$\mathcal{L}(a,b) = \sum_{k,l} c_{kl}\, a^{2k} b^{2l} \tag{4}$$

the amplitudes can be constructed in closed form [9, 10].

As quoted in [1], at the same time when Euler and Kockel were computing the quartic term of the effective Lagrangian, in Moscow a student of Landau's was trying to do the same, but lost this virtual race against Heisenberg's students. Landau reportedly was not happy about this, and next asked his two best students, A. Akhieser and I. Pomeranchuk, to compute the effective four-point interaction in the opposite, high-energy limit. Contrary to the Euler-Kockel calculation, this is a long and hard calculation even by modern standards, and it is remarkable that they were (with Landau's help) able to carry this out successfully. In [11], they showed that

- The total cross section of photon-photon scattering in the high-energy limit behaves like

$$\sigma \sim \alpha^4 (\frac{c}{\omega})^2 \,. \tag{5}$$

- The maximum of the cross section is at $\hbar\omega = mc^2$.

Together with the result of Euler and Kockel for the low-energy limit, this is sufficient to draw a sketch of the cross section in the whole energy range (Fig. 4).

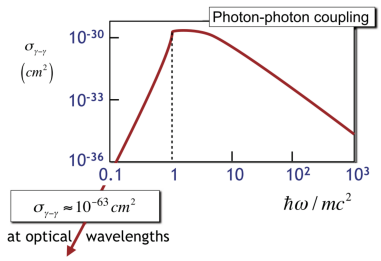


Figure 4. Cross section for photon-photon scattering as a function of energy (from [12]).

2. Experimental measurement of light-by-light scattering

Parallely to these theoretical developments, there was also a long chain of proposals and actual attempts at measuring light-by-light scattering [1]. As early as 1928, S. I. Vavilov performed experiment with light sparks [13], and in 1930 A.L. Hughes and G.E.M. Jauncey tried to scatter rays of sun light [14]. After the Euler-Kockel result it was clear that a measurement of the QED effect at optical frequencies is hopeless, but many more experiments were still done, usually motivated by the exclusion of new physics; see, e.g., [15] and refs. therein.

Thus the direct (although perhaps still not as direct as one might wish) measurement of photon-photon scattering had to wait for a suitable high-energy experiment, and was achieved only in 2017 in lead - lead collisions at the LHC [16] (Fig. 5).

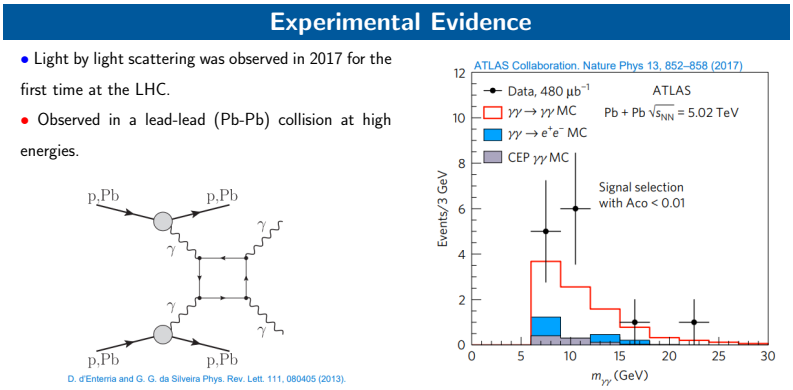


Figure 5. Measurement of LBL scattering by the ATLAS collaboration at LHC. The colliding photons are radiated by lead ions (left panel) with GeV energies. Agreement with the Standard Model predictions is found after subtracting backgrounds such as real electron-positron pair production.

3. Structure of the four-photon amplitude

On the theoretical side, further progress was made in 1951 by R. Karplus and M. Neuman [17, 18] who elucidated the detailed structure of the on-shell four-photon amplitudes. However, this is still not sufficient for all applications; off-shell photons are often called for in processes involving external fields, or if one wishes to use the light-by-light process as a building block for higher-loop amplitudes. Twenty years later, this prompted V. Costantini, B. De Tollis and G. Pistoni [19] to carry out an exhaustive study of the four-photon amplitude with two legs on-shell and two legs off-shell. But a study of the fully off-shell case has, to the best of my knowledge, never been done (even though the technology for the calculation of one-loop four-point off-shell integrals has been available for a number of years, see, e.g., [20, 21, 22, 23, 24]).

Two years ago our collaboration started an effort to close this gap employing the worldline representation of the four-photon amplitudes [25, 26]. The use of this representation is motivated in part by its compactness, in part by its making it easy to integrate out low-energy photon legs. Although a simple consequence of Feynman's path-integral representation of the S-matrix in scalar [27] and spinor QED [28], it was worked out only around 1990 in the wake of developments in QCD and string theory (for a review, see [29]). At the four-photon level, it exists in various slightly different versions that are related by integration-by-parts in (proper-time) parameter space [30, 31, 32, 33, 25]. Let us write down here the particularly compact representation obtained in [25] for the (off-shell) four-photon amplitude in spinor QED:

$$\hat{\Gamma} \;=\; \hat{\Gamma}^{(1)} + \hat{\Gamma}^{(2)} + \hat{\Gamma}^{(3)} + \hat{\Gamma}^{(4)} + \hat{\Gamma}^{(5)}\,, \tag{6}$$

$$
\begin{aligned}
\hat{\Gamma}^{(1)} &= \hat{\Gamma}^{(1)}_{(1234)} T^{(1)}_{(1234)} + \hat{\Gamma}^{(1)}_{(1243)} T^{(1)}_{(1243)} + \hat{\Gamma}^{(1)}_{(1324)} T^{(1)}_{(1324)} \,, \\
\hat{\Gamma}^{(2)} &= \hat{\Gamma}^{(2)}_{(12)(34)} T^{(2)}_{(12)(34)} + \hat{\Gamma}^{(2)}_{(13)(24)} T^{(2)}_{(13)(24)} + \hat{\Gamma}^{(2)}_{(14)(23)} T^{(2)}_{(14)(23)} \,, \\
\hat{\Gamma}^{(3)} &= \sum_{i=1,2,3} \hat{\Gamma}^{(3)}_{(123)i} T^{(3)r_4}_{(123)i} + \sum_{i=2,3,4} \hat{\Gamma}^{(3)}_{(234)i} T^{(3)r_1}_{(234)i} + \sum_{i=3,4,1} \hat{\Gamma}^{(3)}_{(341)i} T^{(3)r_2}_{(341)i} + \sum_{i=4,1,2} \hat{\Gamma}^{(3)}_{(412)i} T^{(3)r_3}_{(412)i} \,, \\
\hat{\Gamma}^{(4)} &= \sum_{i<j} \hat{\Gamma}^{(4)}_{(ij)ii} T^{(4)}_{(ij)ii} + \sum_{i<j} \hat{\Gamma}^{(4)}_{(ij)jj} T^{(4)}_{(ij)jj} \,, \\
\hat{\Gamma}^{(5)} &= \sum_{i<j} \hat{\Gamma}^{(5)}_{(ij)ij} T^{(5)}_{(ij)ij} + \sum_{i<j} \hat{\Gamma}^{(5)}_{(ij)ji} T^{(5)}_{(ij)ji} \,.
\end{aligned}
\tag{7}
$$

Up to permutations, this decomposition of the amplitude involves the following five tensors,

$$
\begin{aligned}
T^{(1)}_{(1234)} &\equiv Z_4(1234) \,, \\
T^{(2)}_{(12)(34)} &\equiv Z_2(12) Z_2(34) \,, \\
T^{(3)r_4}_{(123)i} &\equiv Z_3(123) \frac{r_4 \cdot f_4 \cdot k_i}{r_4 \cdot k_4} \,, \quad i = 1,2,3 \,, \\
T^{(4)}_{(12)ii} &\equiv Z_2(12) \, \frac{k_i \cdot f_3 \cdot f_4 \cdot k_i}{k_3 \cdot k_4} \,, \quad i = 1,2, \\
T^{(5)}_{(12)ij} &\equiv Z_2(12) \, \frac{k_i \cdot f_3 \cdot f_4 \cdot k_j}{k_3 \cdot k_4} \,, \quad (i,j) = (1,2),(2,1).
\end{aligned}
\tag{8}
$$

Their definition involves the photon field-strength tensors $f_i^{\mu\nu} \equiv k_i^\mu \varepsilon_i^\nu - \varepsilon_i^\mu k_i^\nu$ and traces thereof,

$$
Z_n(i_1 i_2 \ldots i_n) \equiv \left(\frac{1}{2}\right)^{\delta_{n2}} \mathrm{tr}\Big(\prod_{j=1}^{n} f_{i_j}\Big) \tag{9}
$$

(r_4 is an arbitrary reference momentum). As shown in [25], this tensor basis is equivalent to the one introduced in [19] based on the QED Ward identity. The coefficient functions in the tensor decomposition involve integrals over the global proper-time T of the fermion in the loop, and over the parameters u_i that indicate the location of the photons along the loop (one of them can be set equal to zero due to the translation invariance in proper-time).

$$
\hat{\Gamma}^{(k)}_{\ldots} = \int_0^\infty \frac{dT}{T} T^{4-\frac{D}{2}} \mathrm{e}^{-m^2 T} \int_0^1 \prod_{i=1}^{4} du_i \, \hat{\gamma}^{(k)}_{\ldots}(\dot{G}_{ij}) \, \mathrm{e}^{\frac{1}{2} T \sum_{i,j=1}^{4} G_{ij} k_i \cdot k_j} \,, \tag{10}
$$

$$
\begin{aligned}
\hat{\gamma}^{(1)}_{(1234)} &= \dot{G}_{12}\dot{G}_{23}\dot{G}_{34}\dot{G}_{41} - G_{F12}G_{F23}G_{F34}G_{F41} \,, \\
\hat{\gamma}^{(2)}_{(12)(34)} &= (\dot{G}_{12}\dot{G}_{21} - G_{F12}G_{F21})(\dot{G}_{34}\dot{G}_{43} - G_{F34}G_{F43}) \,, \\
\hat{\gamma}^{(3)}_{(123)1} &= (\dot{G}_{12}\dot{G}_{23}\dot{G}_{31} - G_{F12}G_{F23}G_{F31})\dot{G}_{41} \,, \\
\hat{\gamma}^{(4)}_{(12)11} &= (\dot{G}_{12}\dot{G}_{21} - G_{F12}G_{F21})\dot{G}_{13}\dot{G}_{41} \,, \\
\hat{\gamma}^{(5)}_{(12)12} &= (\dot{G}_{12}\dot{G}_{21} - G_{F12}G_{F21})\dot{G}_{13}\dot{G}_{42} \,.
\end{aligned}
\tag{11}
$$

The integrands are written in terms of two "worldline Green's functions" G and G_F,

$$G_{ij} = |u_i - u_j| - (u_i - u_j)^2 \,, \tag{12}$$

$$G_{Fij} = \mathrm{sgn}(u_i - u_j) \,. \tag{13}$$

The corresponding representation for scalar QED is simply obtained by deleting in (11) all factors of G_F, and multiplying the whole amplitude by a factor of $-\frac{1}{2}$.

4. Form-factor decomposition of the four-gluon amplitude

The four-photon amplitudes are of special interest in field theory also because they are the prototype for all amplitudes with four gauge bosons. Thus let us digress a little bit and sketch what happens to the above representation in the non-abelian case. If one replaces the photons by gluons, then their color factors force one to fix the ordering of the gluons along the loop. The same integration-by-parts procedure that in the abelian case never produces any boundary terms now will do so. These terms involve color commutators, and have lower-point kinematics. As shown in [34], the above tensor decomposition generalizes to the off-shell four-gluon one-particle irreducible amplitudes (with either a scalar, spinor or gluon loop) yielding a decomposition in terms of 19 tensors. Of those 14 have true four-point kinematics (Fig. 6 (a)), 4 tensors have three-point kinematics (Fig. 6 (b)), and there is one tensor with two-point kinematics (Fig. 6 (c)).

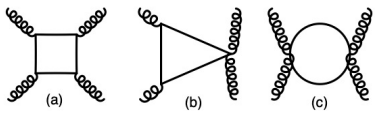


Figure 6. Unpinched, single-pinched and double-pinched kinematics arising in the integration-by-parts procedure of the four-gluon vertex.

Let us list here the 14 tensors that have the full four-point kinematics:

$$\begin{aligned}
T_P^4 &= \mathrm{tr}(f_1 f_2 f_3 f_4)\,, T_{NP}^4 = \mathrm{tr}(f_1 f_3 f_2 f_4)\,,\\
T_P^{22} &= \frac{1}{4}\,\mathrm{tr}(f_1 f_2)\,\mathrm{tr}(f_3 f_4)\,, T_{NP}^{22} = \frac{1}{4}\,\mathrm{tr}(f_1 f_3)\,\mathrm{tr}(f_2 f_4)\,,\\
T_P^3 &= \mathrm{tr}(f_1 f_2 f_3)\varepsilon_4 \cdot k_1\,, T_{NP}^3 = \mathrm{tr}(f_1 f_2 f_3)\varepsilon_4 \cdot k_2\,,\\
T_{\mathrm{quart}}^{2adj} &= \frac{1}{2}\,\mathrm{tr}(f_1 f_2)\varepsilon_3 \cdot k_1 \varepsilon_4 \cdot k_1\,, T_P^{2adj} = \frac{1}{2}\,\mathrm{tr}(f_1 f_2)\varepsilon_3 \cdot k_2 \varepsilon_4 \cdot k_1\,, T_{NP}^{2adj} = \frac{1}{2}\,\mathrm{tr}(f_1 f_2)\varepsilon_3 \cdot k_1 \varepsilon_4 \cdot k_2\,,\\
T_C^{2adj} &= \frac{1}{2}\,\mathrm{tr}(f_1 f_2)\Big(\varepsilon_3 \cdot k_4 \varepsilon_4 \cdot k_1 - \frac{1}{2}\varepsilon_3 \cdot \varepsilon_4 k_4 \cdot k_1\Big)\,,\\
T_Z^{2adj} &= \frac{1}{2}\,\mathrm{tr}(f_1 f_2)\Big(\varepsilon_3 \cdot k_4 \varepsilon_4 \cdot k_2 - \frac{1}{2}\varepsilon_3 \cdot \varepsilon_4 k_4 \cdot k_2\Big)\,,\\
T_{\mathrm{quart}}^{2opp} &= \frac{1}{2}\,\mathrm{tr}(f_1 f_3)\varepsilon_2 \cdot k_1 \varepsilon_4 \cdot k_1\,, T_P^{2opp} = \frac{1}{2}\,\mathrm{tr}(f_1 f_3)\varepsilon_2 \cdot k_3 \varepsilon_4 \cdot k_1\,,\\
T_{NP}^{2opp} &= \frac{1}{2}\,\mathrm{tr}(f_1 f_3)\Big(\varepsilon_2 \cdot k_4 \varepsilon_4 \cdot k_1 - \frac{1}{2}\varepsilon_2 \cdot \varepsilon_4 k_4 \cdot k_1\Big)\,.
\end{aligned}$$

5. Standard model generalizations

In the full standard model, the structure of the four-photon amplitudes is also relevant for the amplitudes with four Z or $W^{\pm}$ bosons. Moreover, the four-photon amplitudes then also have contributions with a $W^{\pm}$-loop, which interestingly dominate the total cross section in the high-energy limit [35].

6. Born-Infeld Electrodynamics

Finally, let us come to more speculative issues. In 1934, just before the work of Euler and Kockel, M. Born and L. Infeld proposed their famous nonlinear generalization of classical Maxwellian electrodynamics [36].

$$\begin{aligned}\mathcal{L}^{BI} &= -b^2\sqrt{-\det\left(\eta_{\mu\nu}+\frac{1}{b}F_{\mu\nu}\right)}+b^2\sqrt{-\det(\eta_{\mu\nu})}\\ &= -b^2\sqrt{1-\frac{2s}{b^2}-\frac{p^2}{b^4}}+b^2 \qquad (14)\end{aligned}$$

where s and p are the two invariants of the Maxwell field, now defined by

$$s \equiv -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} = \frac{1}{2}(E^2-B^2), \qquad (15)$$

$$p \equiv -\frac{1}{4}\tilde{F}^{\mu\nu}F_{\mu\nu} = \vec{E}\cdot\vec{B}\,. \qquad (16)$$

In the limit of large b the Lagrangian (14) reduces to the Maxwell Lagrangian, as can be seen by expanding $\mathcal{L}^{BI}$ in powers of $1/b$:

$$\mathcal{L}^{BI} = s+\frac{1}{2b^2}\left(s^2+p^2\right)+\frac{1}{2b^4}\left(s^3+sp^2\right)+\ldots. \qquad (17)$$

A main point of this theory is that the classical self-energy of the electron becomes finite, so that one can attempt to identify it with its rest mass energy, which thus becomes calculable. This is certainly one of the most attractive ideas that have ever been put forward for the origin of mass, and very different from standard QED, where the electron self-energy has to be made finite by renormalization, in the process of which one loses control over its numerical value, which becomes a quantity to be determined by experiment. It leads to the following numerical value of b,

$$b = 1.2\times 10^{20}\frac{V}{m}\,. \qquad (18)$$

Euler and Kockel in their above-cited paper [5] take up this idea and extend it to a calculation of the fine structure constant, simply by equating the quartic terms of their quantum effective Lagrangian with the quartic terms of the classical Born-Infeld Lagrangian (Fig. 7).

Es ist von Interesse, diesen Zusatz, der durch die quantentheoretische Möglichkeit der Paarerzeugung zur MAXWELLschen Energie hinzukommt, mit dem von BORN[2] aus Überlegungen im Rahmen der klassischen Theorie vorgeschlagenen zu vergleichen, deren erstes Entwicklungsglied heißt:

$$\frac{-(1{,}2361)^4}{32\,\pi}\cdot\frac{1}{E_0^2}\int[(\mathfrak{B}^2-\mathfrak{E}^2)^2+4(\mathfrak{B}\,\mathfrak{E})^2]\,dV\,.$$

Abgesehen von der Tatsache, daß das Verhältnis der Koeffizienten der beiden Zusatzglieder bei BORN 1 : 4, bei uns 1 : 7 ist, unterscheiden sich die beiden Ausdrücke um den Faktor:

$$\frac{4}{45\,\pi\cdot(1{,}2361)^4}\cdot\frac{\hbar c}{e^2}\,.$$

Wegen des tatsächlichen Werts der SOMMERFELDschen Feinstrukturkonstanten ist der numerische Wert dieses Faktors $\sim 1{,}7$, und es ist bemerkenswert, daß die quantentheoretische Abänderung der MAXWELLschen Gleichungen jedenfalls von der Größenordnung ist, die nach klassischer Vorstellung auf Grund der Selbstenergie zu erwarten wäre.

Figure 7. Paragraph from [5] on equating the quartic terms of the Euler-Heisenberg and Born-Infeld Lagrangians.

This actually cannot be done exactly, as they mention, since these quartic parts contain two independent terms whose prefactors in the Euler-Heisenberg case come in a ratio of 1:7 and in the Born-Infeld case in a ratio of 1:4. Nevertheless, one gets the correct order of magnitude of the fine structure constant, which is already very remarkable.

However, Euler and Kockel did not provide any justification for equating these two Lagrangians, and were roundly criticized by W. Pauli in a letter to V. Weisskopf on September 27, 1935 (as quoted in [1]): "In my view, in their published note Euler and Kockel have put much too strong emphasis on the comparison with this monstrosity of a pseudo-theory".

It is not known whether Heisenberg got to know about this criticism, but perhaps it is in response to it that, in his paper with Euler, they included the paragraph shown in Fig. 8.

The results derived in the second section are very similar to Born's[9] derivation of Maxwell equations. Also Born obtains a more complicated function of the two invariants, $\mathfrak{E}^2 - \mathfrak{B}^2$ and $(\mathfrak{E}\mathfrak{B})^2$ in addition to the classical Lagrangian $\mathfrak{E}^2 - \mathfrak{B}^2$. Incidentally, because of the actual value of $\frac{e^2}{\hbar c}$, this function agrees with (43) in the order of magnitude of the lowest expansion terms (compare 3). Though it is also important to emphasize the differences of both results. Born has chosen the modified Maxwell equation as the starting point of the theory, whereas this change in the Dirac theory is a really indirect consequence of the virtual possibility of the pair creation. In addition, Dirac's theory predicts also terms, which contain higher powers of the field strengths (compare 1, 2). Thus, especially the question of the self-energy of the electrons cannot be decided only with the help of these changes. The result of Born's theory imply that the changes of the Maxwell equations calculated here suffice to remove difficulties with the infinite self energy is an important indication of the further development of the theory.

Figure 8. Paragraph from [6] about the distinction between the Euler-Heisenberg and Born-Infeld Lagrangians.

In modern terminology, they are making here a clear distinction between the concept of a fundamental (Born-Infeld) and an effective (Euler-Heisenberg) Lagrangian.

After the development of QFT, the Born-Infeld theory as a fundamental non-linear one of electrodynamics fell into disregard for a few decades since it leads, of course, to a non-renormalizable theory after quantization. However, in the eighties it enjoyed a striking comeback when E. S. Fradkin and A. A. Tseytlin [37] discovered that it is generated by open string theory as a tree-level effective Lagrangian!

Thus nowadays it is often considered more generally as an effective Lagrangian generated by some unknown new physics, with a parameter b to be determined by experiment. However, I believe that two warnings are in order:

First, it is very common in the recent literature to consider LBL scattering as an experimental test of "Euler-Heisenberg vs. Born-Infeld" (see, e.g., [38, 39]). Here it must be kept in mind that, differently from the times of Euler and Kockel, today we do not have any more the option of doubting the existence of the quartic (or any other) Euler-Heisenberg terms, since they follow directly from the existence of the Dirac vertex and the well-tested principles of QFT. What is still viable is to assume a small four-photon vertex of unknown origin in addition to the Euler-Heisenberg one (Fig. 9).

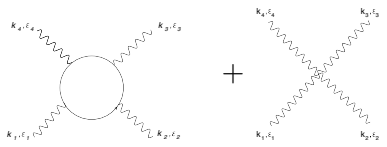


Figure 9. Adding the QED amplitude (left) and a four-photon vertex of unknown origin (right).

But since experiments will not resolve between the two cases, then one also needs to include the interference term. Assuming that the quartic vertex is of Born-Infeld type, the total cross section becomes [40]

$$\sigma = \left(\frac{973\ \alpha^4}{10125\ m^8} + \frac{77\ \alpha^2}{225\ b^2\ m^4} + \frac{7}{20\ b^4} \right) \frac{\omega^6}{\pi}\,. \tag{19}$$

Here the leftmost term in brackets is the pure Euler-Heisenberg contribution, the rightmost one the pure Born-Infeld one, and the middle term comes from interference between the two.

Second, when using the Born-Infeld Lagrangian for any modelling one has to face a basic quandary: considering it as a fundamental Lagrangian will lead to the loss of renormalizability and, at the loop level, to severe IR divergences [41]. Taking it as an effective Lagrangian, one must be aware that there will unavoidably be derivative corrections (terms involving derivatives of $F_{\mu\nu}$). For the string effective action such corrections have been calculated [42, 43], but in general they will be essentially unkown, and negligible only in the low-energy limit.

In any case, one question remains, why did Euler and Kockel get so close to the actual value of α? From a modern perspective, there seems to be no intrinsic relation between standard QED and Born-Infeld theory, and no reason why that calculation could not have failed by many orders of magnitude. Did they just get lucky?

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