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A variance-reduction strategy for the sensitivity of β_{eff}

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Abstract. The Monte Carlo computation of the GPT-based sensitivity of the effective delayed neutron fraction β_{eff} to nuclear data proves to be quite difficult to converge due to the small amount of delayed neutrons that are sampled in k-eigenvalue calculations. This paper describes a variance-reduction method aimed at efficiently computing the sensitivity coefficients of β_{eff}, reducing the associated Monte Carlo uncertainty and increasing the Figure Of Merit. This variance-reduction technique allows also computing the sensitivities of β^j_{eff} for a specific precursor family j. Verification and performance evaluation are achieved using simple configurations admitting analytical solutions and several continuous-energy benchmarks.

1 Introduction

Over the past fifteen years, sensitivity analysis and uncertainty propagation for k-eigenvalue problems have become a standard feature of production-level Monte Carlo codes using continuous-energy [1]. In this framework, various implementations of Generalized Perturbation Theory (GPT) have been recently developed and tested [2–6].

In TRIPOLI-4®, the Monte Carlo code developed at CEA, we implemented a GPT method that can be applied to ratios of reaction rates and ratios of adjoint-weighted quantities; these latter encompass effective kinetics parameters such as the neutron generation time Λ_{eff}, the prompt neutron lifetime ℓ_{eff}, and the delayed neutron fraction β_{eff}. In a previous work, we observed that computing the sensitivity of β_{eff} with respect to nuclear data was quite challenging, due to extremely large statistical uncertainties, especially for (but not limited to) scattering cross sections [7]. Indeed, our preliminary simulation results exhibited uncertainties up to tens of percent in the very simple Jezebel benchmark (a bare homogeneous Pu sphere), despite a total of 400 millions of sampled neutron histories. A key reason for this behavior is the small amount of delayed neutrons that are sampled in k-eigenvalue calculations: similar issues have been reported for an independent implementation in the Serpent code [3].

In view of these considerations, in this paper we propose a variance-reduction technique that allows efficiently computing the sensitivity of β_{eff} to nuclear data. In order to assess the performance of the new method, we will use the standard Figure Of Merit (FOM) metric, defined as FOM= $1/\sigma^2 T$, where σ^2 is the variance and T is the computation time.

This paper is organised as follows: Sec. 2 will briefly recall the GPT formalism needed to estimate the sensitivity of β_{eff}. Section 3 describes the variance-reduction algorithm. In Sec. 4

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we will then verify the method and quantify the FOM gain on a few relevant benchmarks. Finally, in Sec. 5 we will display preliminary results concerning a generalization to the case of the sensitivity of β^j_{eff}, the effective delayed neutron fraction of precursor family j. Conclusions will be drawn in Sec. 6.

2 Sensitivity of the effective delayed neutron fraction

The effective delayed neutron fraction β_{eff} is defined as

$$\beta_{\text{eff}} = \frac{\langle \varphi^\dagger F_d \varphi \rangle}{\langle \varphi^\dagger F \varphi \rangle} = \frac{R_1^*}{R_2^*}, \tag{1}$$

where F is the total (prompt plus delayed) fission production operator, F_d the delayed fission production, φ the forward fundamental mode of the k-eigenvalue problem, and $\varphi^\dagger$ the associated adjoint eigenmode. As such, the computation of the numerator R_1^* and the denominator R_2^* of β_{eff} requires the simultaneous estimation of the forward and adjoint eigenmode; this latter can be achieved using the neutron progeny in order to infer the importance at a later generation [8].

The sensitivity S_α of β_{eff} with respect to a nuclear data parameter α can be decomposed as

$$S_\alpha(\beta_{\text{eff}}) = \frac{\alpha}{\beta_{\text{eff}}} \frac{\partial \beta_{\text{eff}}}{\partial \alpha} = S_\alpha(R_1^*) - S_\alpha(R_2^*). \tag{2}$$

In TRIPOLI-4®, the algorithm that allows estimating each term $S_\alpha(R_i^*)$, $i = 1, 2$, relies on the Differential Operator Sampling (DOS) method, coupled to a super-history power iteration scheme [6]. For each 'outer' cycle, a neutron and its progeny are simulated over a given number of 'inner' generations; only fission neutrons of the last inner generation are pushed to the next outer cycle. Super-history considerably reduces the memory burden of sensitivities calculations compared to the Iterated Fission Probability (IFP) implementation, at the expense of an increase in the total simulation time. Following the DOS strategy, the terms $S_\alpha(R_i^*)$ are estimated along the neutron progeny using

$$S_\alpha(R_i^*) = \frac{1}{R_i^*} \sum_{n \in g} \sum_{c} \sum_{m \in d^g_{n,c}} p^g_{n,c} \xi^g_{n,c}(R_i^*) \xi^{g+L}_{m,c}(k) \left(\sum_{\ell = g-L}^{g+L} \mu^{0,\ell}_{n_f,c_f} + \frac{\xi^{\ell,\prime}_{n_f,c_f}(k)}{\xi^{\ell}_{n_f,c_f}(k)} \right), \tag{3}$$

where g denotes the inner generation index within an outer cycle, n is the neutron history identifier within a generation g, c is the collision index within a neutron history, $d^g_{n,c}$ denotes the progeny of a fission event induced by neutron n of generation g at collision site c, $\xi^{g+L}_{m,c}(k)$ is the random contribution to production rate ($\nu_t \sigma_f$) at the collision point of a neutron belonging to the generation $g + L$ resulting from the progeny of the collision c which happened at generation g, μ^0 corresponds to the derivatives along the neutron history events, $p^g_{n,c}$ refers to the neutron's life probability from birth to collision c, and $\xi^g_{n,c}(R_i^*)$ is the random contribution of the neutron n to the tally R_i^*. The sum from $g - L$ to $g + L$ allows estimating the effect of the nuclear data perturbation on the ancestors of the neutron ($\ell < g$: fission source effect), on the current neutron history ($\ell = g$) and on the neutron progeny ($\ell > g$: neutron importance) [6]. Recommended values for L are case-dependent, but typically lie around 10. For an illustration, see Fig. 1. This scheme is compatible with standard non-analog sampling methods, such as implicit absorption and forced fission.

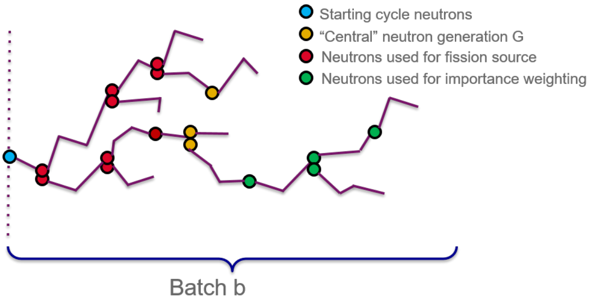

Figure 1. Super-history scheme of the GPT calculations involving adjoint-weighted tallies. Blue particles denote starting neutrons of a cycle; red particles denote the inner generations used for the fission source perturbation estimation (the blue neutron is also used for this estimation); yellow particles denote the central generation of the super-history scheme at which the generation time and the delayed neutron fraction contribution are computed; green particles denote the progeny of the yellow neutron used for evaluating the importance estimation and its derivatives.

3 A variance-reduction scheme for the sensitivity of β_{eff}

For the computation of the sensitivities of the effective delayed neutron fraction with respect to nuclear data, the determination of the contribution to the delayed neutron fraction occurs in the 'central' generation inside the super-history scheme (neutrons marked in yellow in Fig. 1). In TRIPOLI-4®, the number of delayed neutrons per fission event is sampled as $n_{\text{delayed}} = \lfloor \nu_d \sigma_f / \sigma_t + \xi \rfloor$, with σ_f the fission cross section, σ_t the total cross section, ν_t, ν_p, ν_d respectively the total, prompt and delayed fission yield and ξ a random number uniformly distributed in (0, 1), which means that n_{delayed} is typically small, even though the fission event is forced whenever the collided nucleus is fissile ($\sigma_f > 0$). Since $\nu_p \gg \nu_d$, a discrepancy arises between the sampled population of prompt and delayed neutrons. This is crucial for the GPT algorithm detailed above and is ultimately responsible for the aforementioned difficulty in obtaining a reasonable statistical convergence for $S_\alpha(\beta_{\text{eff}})$.

Our approach to circumvent this issue is to artificially boost the number of sampled delayed neutrons at each fission event, by taking

$$n_{\text{delayed}} = \lfloor \nu_t \sigma_f / \sigma_t + \xi \rfloor, \tag{4}$$

i.e., replacing ν_d by ν_t. Since $\nu_t \simeq \nu_p$, each fission event will lead to a comparable number of sampled prompt and delayed neutrons. To ensure a fair (unbiased) game, we need then to adjust the statistical weights of the sampled delayed neutrons by multiplying them by the delayed neutron fraction $\beta = \nu_d / \nu_t$ of the collided nucleus.

Care must be taken, however, because the forced delayed neutrons (sampled at generation g) will only contribute to the sensitivity of β_{eff} if their progeny survive over L generations (the neutron importance is estimated at generation $g + L$). The main issue is related to Russian roulette, since the typical value of β is much lower than the 'standard' values of the roulette threshold w_{rr}: in TRIPOLI-4®, e.g., $w_{\text{rr}} = 0.8$, and in MCNP6.3® $w_{\text{rr}} = 0.25$. If the threshold is not adjusted accordingly, very few boosted delayed neutrons will survive along the L successive generations.

In order to counterbalance the effect of the weight correction factor, we modify the naive implementation as follows. The forced delayed neutrons are assigned a flag at their birth; their weight is corrected after the fission (birth) event, but the weight correction factor β is stored and used as the central (and survival) weight for the application of the Russian roulette for each subsequent collision event of the neutron. This 'forced delayed' flag is passed on to their progeny in order to apply the same Russian roulette strategy until generation $g + L$ of the super-history scheme. The forced delayed fission strategy is only applied in the 'central' generation as this is the generation where we estimate the delayed neutron fission contribution.

4 Verification and validation

4.1 A simple benchmark with analytical solutions

As a first verification of our variance-reduction scheme, we used a a simple benchmark consisting of an infinite medium with two energy groups and two precursor families, for which exact solutions for the sensitivity of β_{eff} have been derived [6]. Fission occurs only in the thermal group, upscattering is neglected, and prompt neutrons are emitted only in the fast group. The nuclear data for this benchmark are recalled in Tab. 1; the delayed neutron fractions β_j for the two precursor families have been set so that $\beta = \sum_j \beta_j = 700 \times 10^{-5}$. The capture cross section has been adjusted in order to have an almost critical configuration. Simulations were run with 100 000 neutrons per cycle and 1000 active cycles.

Table 1. Nuclear data for the two-group benchmark. Symbols $\chi_{j,g}$ denote delayed fission spectra for family j, in energy group g. The delayed neutron fractions for the two precursor families (denoted a and b afterwards) are set to $\beta_a = 520 \times 10^{-5}$ and $\beta_b = 180 \times 10^{-5}$, with $\beta = \sum_j \beta_j = 700 \times 10^{-5}$.

Data	Fast group ($g = 1$)	Thermal group ($g = 2$)
$\Sigma_{c,g}$	1.0	0.91
$\Sigma_{f,g}$	0.0	1.5
$\Sigma_{s,g\to 1}$	0.5	0.0
$\Sigma_{s,g\to 2}$	0.5	1.0
$\nu_{f,g}$	0.0	24/5
$\chi_{a,g}$	0.75	0.25
$\chi_{b,g}$	0.5	0.5

The computed k_{eff}, β_{eff} and Λ_{eff} are compared to the reference solutions in Tab. 2. Monte Carlo estimates with and without the variance-reduction method described in Sec. 3 are in excellent agreement with each other. A large increase of the FOM (by a factor ~ 30) is observed for the computation of β_{eff}; the FOM of k_{eff} and Λ_{eff} on the contrary drops by about 25%. This behaviour is due to the fact that the increase in simulation time (more particles are simulated because of the variance-reduction method) is largely compensated by the decrease of the standard deviation on β_{eff} by two orders of magnitude; the decrease on k_{eff} and Λ_{eff} is only about one order of magnitude and hardly compensates the increase in simulation time.

Table 2. Comparison of effective multiplication factor, effective delayed neutron fraction and effective neutron generation time between analytical solutions, standard Monte Carlo (MC) and Monte Carlo with the variance-reduction technique for β_{eff} (MC$^{\text{vr}}$).

	Analytical	MC $\pm\ \sigma(\%)$	MC$^{\text{vr}}$ $\pm\ \sigma(\%)$	FOM gain
k_{eff}	1.00021	1.00012 ± 0.019	1.00019 ± 0.003	0.74
β_{eff}	0.01133	0.01125 ± 1.393	0.01133 ± 0.031	31.85
Λ_{eff}	0.14919	0.14922 ± 0.116	0.14918 ± 0.017	0.74

In Tab. 3 the sensitivity of β_{eff} computed using Monte Carlo estimates with and without the variance-reduction method are compared to analytical solutions: an excellent statistical agreement is found. The variance-reduction method makes the relative standard deviation drop from ~8.5% to ~0.2% and the FOM correspondingly increases by a factor ~30, similar to the gain obtained for mean value estimate of β_{eff}. The results obtained in this benchmark configuration suggest that the variance-reduction scheme is very effective.

Table 3. Sensitivities of β_{eff} with respect to two nuclear data parameters α: comparison of standard Monte Carlo (MC) and Monte Carlo with the variance-reduction technique (MC$^{\text{vr}}$) to analytic solutions.

	Analytic	MC		MC$^{\text{vr}}$		
α		mean ± σ(%)	MC/analytic	mean ± σ(%)	MC$^{\text{vr}}$/analytic	FOM gain
$\Sigma_{c,1}$	0.3803	0.3683 ± 8.73	0.97	0.3792 ± 0.20	1.00	30.7
$\Sigma_{s,1}$	-0.3803	-0.3626 ± 8.58	0.95	-0.3817 ± 0.19	1.00	31.2

4.2 Continuous-energy benchmarks

In order to assess the performance of the variance-reduction method on more realistic configurations, we have selected three continuous-energy benchmarks. The first configuration is PU-MET-FAST-001 (4.5 at.%^{240}Pu, 1.02 wt.% Ga), also known as Jezebel, taken from the ICSBEP handbook [9]. This configuration consists of a bare sphere of metal plutonium with a radius of 6.38493 cm. TRIPOLI-4® simulations have been performed using 2 000 active cycles of 200 000 neutrons each. Five neutron generations have been used to account for the source effect and five additional generations for the importance-weighting effects: previous investigations have shown that this choice is satisfactory, due to the small size of Jezebel [6].

In Tab. 4 we present the comparison between TRIPOLI-4® simulation results (with and without the variance-reduction technique) and those reported in literature for SUSD3D [10]. These results show a large increase of the FOM by a factor 60 to 220, depending on the considered nuclear data: this suggests that the variance-reduction technique is again very effective. Furthermore, for the the sensitivities to the fission, elastic and inelastic scattering cross sections, observe that in the standard implementation (i.e., without the variance-reduction method) the estimated values were basically unreliable, since statistical uncertainties were in the range from 10% to 35%. The variance-reduction method dramatically affects the statistical uncertainty for these quantities, and allows obtaining reliable Monte Carlo estimates. The very large gain in terms of FOM most probably stems from the fact that the effective delayed neutron fraction of Jezebel is very low, say $\beta_{\text{eff}} \sim 185$ pcm, which means that replacing ν_d by ν_t in the forced sampling of the delayed neutron might lead to a huge difference in the number of simulated particles and in the associated variance.

We examine next the HEU-MET-FAST-028 benchmark, also known as FLATTOP-25, which is a highly-enriched (93.24 wt.%) uranium sphere with a radius of 6.1156 cm, reflected by a sphere of natural uranium with a radius of 24.1242 cm. For this configuration, we used 100 000 neutrons per cycle and 5 000 active cycles. Five neutron generations have been used to account for the source effect, and five additional generations for the adjoint-flux-weighting effects. This is again coherent with previous investigations [6].

Table 5 displays the comparison between TRIPOLI-4® simulation results (with and without the variance-reduction technique) and those reported in literature for SUSD3D. These findings show an increase of the FOM by a factor ranging from ~ 30 to ~ 50 with the use of the variance-reduction technique. In particular, the use of the variance-reduction method leads to a relevant decrease in the statistical uncertainty for the sensitivity coefficients of β_{eff} with respect

Table 4. β_{eff} and sensitivity of β_{eff} for Jezebel. T4 denotes the standard TRIPOLI-4® calculations, while T4$^{\text{vr}}$ denotes the use of the variance-reduction technique in TRIPOLI-4®.

Data	SUSD3D	T4	σ(%)	T4$^{\text{vr}}$	σ(%)	T4$^{\text{vr}}$/SUSD3D	FOM gain
β_{eff} [pcm]	185	184	0.3	185	0.02	1.00	164
^{239}Pu σ_f	-0.014	-0.013	18.51	-0.013	1.10	0.93	137
^{239}Pu σ_{el}	0.079	0.071	10.74	0.080	0.51	1.01	213
^{239}Pu σ_{in}	0.009	0.013	33.46	0.008	2.85	0.89	66
^{239}Pu $\sigma_{n,g}$	-0.022	-0.022	1.25	-0.022	0.06	1.00	195
^{239}Pu ν_d	0.948	0.949	0.08	0.949	0.00	1.00	221
^{239}Pu ν_p	-0.947	-0.947	0.15	-0.946	0.01	1.00	137

to the ^{235}U elastic and inelastic scattering and the ^{238}U elastic scattering: they drop respectively from 29.0% to 4%, from 22.6% to 2.8% and from 17.7% to 1.8%. This contributes again to enhancing the reliability of Monte Carlo estimates for the calculation of the sensibility of β_{eff}. The FOM gain is overall lower than the one observed for Jezebel, which might stem from the fact that the effective delayed neutron fraction of FLATTOP-25 is considerably larger, say $\beta_{\text{eff}} \sim 690$ pcm.

Table 5. β_{eff} and sensitivity of β_{eff} for FLATTOP-25. T4 denotes the standard TRIPOLI-4® calculations, while T4$^{\text{vr}}$ denotes the use of the variance-reduction technique in TRIPOLI-4®

Data	SUSD3D	T4	σ(%)	T4$^{\text{vr}}$	σ(%)	T4$^{\text{vr}}$ /SUSD3D	FOM gain
β_{eff} [pcm]	688	689	0.2	690	0.02	1.00	41.8
^{235}U σ_f	-0.0585	-0.05203	3.74	-0.0540	0.44	0.92	40.8
^{235}U σ_{el}	0.0158	0.01903	28.97	0.0163	3.97	1.03	30.6
^{235}U σ_{in}	-0.0141	-0.01224	22.56	-0.0119	2.84	0.85	36.2
^{235}U $\sigma_{n,g}$	-0.033	-0.03536	1.12	-0.0359	0.13	1.09	42.0
^{238}U σ_f	0.0282	0.02068	5.53	0.0203	0.72	0.72	34.3
^{238}U σ_{el}	0.0472	0.04296	17.72	0.0492	1.84	1.04	53.5
^{238}U σ_{in}	-0.0512	-0.04755	6.76	-0.0467	0.83	0.91	37.9
^{238}U $\sigma_{n,g}$	-0.0133	-0.01501	5.12	-0.0146	0.63	1.10	38.0
^{235}U ν_d	0.836	0.836	0.08	0.837	0.01	1.00	36.5
^{235}U ν_p	-0.843	-0.833	0.13	-0.834	0.02	0.99	33.5
^{238}U ν_d	0.153	0.151	0.38	0.152	0.05	0.99	36.0
^{238}U ν_p	-0.140	-0.149	0.70	-0.149	0.09	1.06	32.7

Finally, in order to extend our verification to a more complex system, we have performed a calculation using the EPICURE configuration of the EOLE research reactor, formerly operated at CEA [11], which was previously considered for the analysis of the preliminary implementation of the GPT capabilities in TRIPOLI-4®. The EPICURE UM17x17 configuration is a thermal reactor composed of 289 MOX fuel pins and 1351 UOX fuel pins: a central 17×17 MOX-7% lattice, surrounded by 3.7% U-235 enriched UO2 pins with a lattice pitch of 1.26 cm. The active zone has a height of 80 cm, and the total height of the geometry included in the numerical model is around 1.2 m, which encompasses grids, vessel, water and air. Radial and axial cuts of the TRIPOLI-4® model of EPICURE are illustrated in Fig. 2. For the purpose of verification, we compared TRIPOLI-4® to the APOLLO3® deterministic code [12], which can compute the sensitivity effective delayed neutron fraction [13, 14].

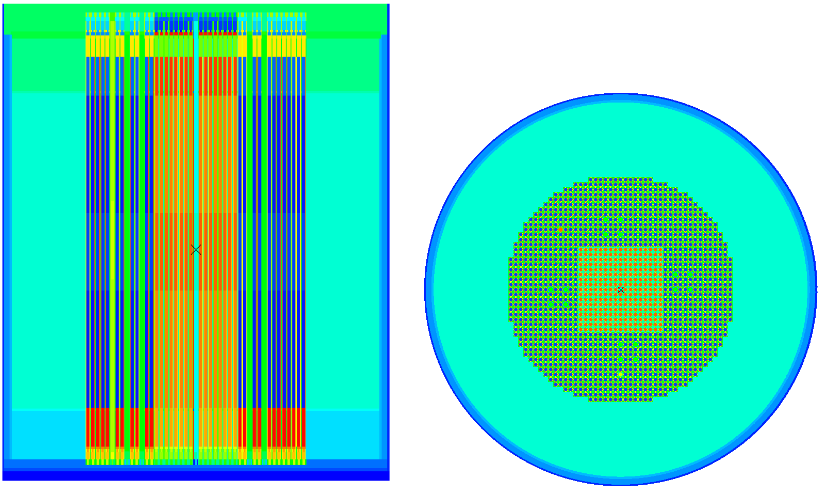

Figure 2. Axial and radial cuts of the EPICURE numerical model for TRIPOLI-4®. Left: x cross section at $x = 0$. Right: z cross section at $z = 0$.

Table 6 displays the sensitivity of β_{eff} computed with APOLLO3® and TRIPOLI-4® with and without the variance-reduction scheme. Overall, a general agreement is found between APOLLO3® and TRIPOLI-4®. The use of the variance-reduction method in TRIPOLI-4® allows reducing the statistical uncertainty by a factor 10 and obtaining FOM gains ranging from 25 to 60. Again, a highly beneficial effect is observed for the sensitivity to scattering cross sections, whose estimates become statistically reliable thanks to the variance-reduction method.

Table 6. Sensitivity of β_{eff} for EPICURE. T4 denotes the standard TRIPOLI-4® calculation,s while T4$^{\text{vr}}$ denotes the use of the variance-reduction technique in TRIPOLI-4®.

Data	AP3	T4	σ(%)	T4$^{\text{vr}}$	σ(%)	T4$^{\text{vr}}$/AP3	FOM gain
^{239}Pu σ_f	-0.1517	-0.1388	1.43	-0.1374	0.17	0.91	44.9
^{235}U σ_f	0.107	0.0967	3.44	0.0962	0.39	0.90	49.4
^{1}H σ_{el}	-0.0852	-0.0766	-31.6	-0.0767	-3.68	0.90	46.8
^{238}U σ_f	0.0647	0.0626	2.07	0.0642	0.25	0.99	43.5
^{239}Pu $\sigma_{n,g}$	0.0646	0.0647	1.51	0.0636	0.18	0.98	44.7
^{238}U σ_{in}	-0.033	-0.0328	8.18	-0.0336	0.91	1.02	51.3
^{235}U $\sigma_{n,g}$	-0.0332	-0.0321	2.25	-0.0318	0.26	0.96	47.5
^{240}Pu $\sigma_{n,g}$	0.0448	0.0313	2.07	0.0299	0.26	0.67	40.2
^{16}O σ_{el}	-0.024	-0.0209	50.6	-0.0228	5.47	0.95	54.3
^{1}H $\sigma_{n,g}$	-0.0146	-0.0139	4.97	-0.0135	0.57	0.92	48.3
^{238}U $\sigma_{n,g}$	-0.0029	-0.0072	17.77	-0.0052	2.81	1.79	25.4
^{240}Pu σ_f	-0.0033	-0.0029	7.41	-0.0032	0.82	0.97	51.8
^{238}U σ_{el}	-0.0058	-0.0019	427.71	-0.0028	34.9	0.48	95.4
^{238}U $\sigma_{n,2n}$	-0.0017	-0.0014	17.37	-0.0016	1.78	0.94	60.5
^{235}U σ_{in}	-0.0006	-0.0006	67.46	-0.0005	9.02	0.83	35.5

These investigations suggest that the implementation of the novel variance-reduction scheme for the sensitivities of β_{eff} with respect to nuclear data is correctly working in TRIPOLI-

4®. The estimates obtained in the modified version of the GPT capability are consistent with published results and with the values obtained from the standard version of the GPT calculations in TRIPOLI-4® (i.e., without the variance-reduction method). In all tested cases, a significant decrease in the statistical uncertainty and a significant increase in the FOM have been observed when using the variance-reduction scheme, which supports its effectiveness for this class of Monte Carlo simulations.

5 Sensitivities of β_{eff}^{j} for a specific precursor family j

Prompted by the successful implementation of the variance-reduction strategy for the total effective delayed neutron fraction β_{eff}^{j}, we implemented a feature for computing the sensitivity of the family-wise effective delayed neutron fractions β_{eff}^{j} with respect to nuclear data. The extension from computing sensitivity of the total effective delayed neutron fraction β_{eff} to family-wise β_{eff}^{j} consists of filtering out the random contribution according to the sampled family j for the delayed neutrons produced in the 'central' generation.

In Tab. 7 and Tab. 8 we display the sensitivity of β_{eff}^{j} with respect to the cross sections of ^{235}U and ^{238}U. Statistical uncertainties are overall very low, except for some sensitivities that come with extremely small values, such as the sensitivity of $\beta_{\text{eff}}^{j=2}$ with respect to the fission cross section of ^{235}U, or the sensitivity of $\beta_{\text{eff}}^{j=6}$ with respect to the elastic cross section of ^{235}U.

Table 7. FLATTOP-25 benchmark: sensitivities of β_{eff}^{j} per precursor family j with respect to the cross sections of ^{235}U. Variance reduction is activated.

	β_{eff}^{j} [pcm] $\pm\sigma(\%)$	^{235}U σ_f $\pm\sigma(\%)$	^{235}U σ_{el} $\pm\sigma(\%)$	^{235}U σ_{in} $\pm\sigma(\%)$	^{235}U $\sigma_{n,g}$ $\pm\sigma(\%)$
$j = 1$	22 ± 0.08	0.0246 ± 3.75	0.0396 ± 6.66	0.0141 ± 9.20	-0.0480 ± 0.40
$j = 2$	117 ± 0.04	0.0002 ± 268.70	0.0258 ± 4.76	0.0082 ± 7.72	-0.0394 ± 0.23
$j = 3$	115 ± 0.04	-0.0218 ± 2.11	0.0260 ± 4.92	0.0011 ± 56.21	-0.0404 ± 0.23
$j = 4$	265 ± 0.03	-0.0524 ± 0.64	0.0146 ± 6.16	-0.0109 ± 4.21	-0.0328 ± 0.20
$j = 5$	120 ± 0.04	-0.1265 ± 0.39	0.0025 ± 52.35	-0.0403 ± 1.63	-0.0336 ± 0.27
$j = 6$	50 ± 0.06	-0.1226 ± 0.58	0.0016 ± 122.06	-0.0386 ± 2.46	-0.0341 ± 0.40
Total	690 ± 0.02	-0.0540 ± 0.44	0.0163 ± 3.97	-0.0119 ± 2.84	-0.0359 ± 0.13

Table 8. FLATTOP-25 benchmark: sensitivities of β_{eff}^{j} per precursor family j with respect to the cross sections of ^{238}U. Variance reduction is activated.

	^{238}U σ_f $\pm\sigma(\%)$	^{238}U σ_{el} $\pm\sigma(\%)$	^{238}U σ_{in} $\pm\sigma(\%)$	^{238}U $\sigma_{n,g}$ $\pm\sigma(\%)$
$j = 1$	-0.0785 ± 0.68	0.0440 ± 8.21	-0.0036 ± 42.47	-0.0064 ± 5.79
$j = 2$	-0.0398 ± 0.67	0.0501 ± 3.32	-0.0165 ± 4.50	-0.0087 ± 1.97
$j = 3$	-0.0188 ± 1.48	0.0532 ± 3.31	-0.0311 ± 2.39	-0.0116 ± 1.56
$j = 4$	0.0224 ± 0.92	0.0487 ± 2.59	-0.0444 ± 1.20	-0.0145 ± 0.88
$j = 5$	0.0987 ± 0.33	0.0485 ± 3.72	-0.0884 ± 0.89	-0.0220 ± 0.86
$j = 6$	0.0948 ± 0.49	0.0441 ± 6.17	-0.0849 ± 1.37	-0.0217 ± 1.27
Total	0.0203 ± 0.72	0.0492 ± 1.84	-0.0467 ± 0.83	-0.0146 ± 0.63

To the best of our knowledge, published results concerning the sensitivity of family-wise β_{eff}^{j} are not available. Thus, in order to validate our implementation, we make use of the relation

$$S_\alpha(\beta_{\text{eff}}) = \sum_j \frac{\beta^j_{\text{eff}}}{\beta_{\text{eff}}} \times S_\alpha(\beta^j_{\text{eff}}) \tag{5}$$

between the sensitivity of the total β_{eff} and family-wise β^j_{eff}. In Tab. 9 we display the comparison of the sensitivity of β_{eff} computed directly by TRIPOLI-4® and reconstructed from family-wise values through Eq. (5). The maximum relative difference attains 1% for the elastic cross section of ^{235}U; most of other cases lie below 0.4%, which supports the consistency of our algorithm.

Table 9. Sensitivity of β_{eff} computed directly using TRIPOLI-4® and reconstructed from family-wise sensitivities.

	S(β_{eff})	σ(%)	Reconstructed	ratio
^{235}U σ_f	-0.0540	0.44	-0.0539	0.997
^{235}U σ_{el}	0.0163	3.97	0.0161	0.990
^{235}U σ_{in}	-0.0119	2.84	-0.0120	1.004
^{235}U $\sigma_{n,g}$	-0.0359	0.13	-0.0359	0.999
^{238}U σ_f	0.0203	0.72	0.0202	0.997
^{238}U σ_{el}	0.0492	1.84	0.0492	0.999
^{238}U σ_{in}	-0.0467	0.83	-0.0467	1.000
^{238}U $\sigma_{n,g}$	-0.0146	0.63	-0.0146	0.999

6 Conclusions

In this work, we proposed a variance-reduction method to improve the computation of the sensitivity of the effective delayed neutron fraction, within the framework of the GPT capabilities of TRIPOLI-4®. This technique relies on a forced sampling approach of the delayed neutrons at a specific generation of the super-history and a subsequent modification of the Russian roulette parameters in order to boost the progeny sampled from this forced event.

Verification has been performed on simple configurations allowing for analytical solutions and more realistic continuous-energy benchmarks. The performance of the variance-reduction strategy was extensively assessed. Significant improvements in terms of FOM were observed, with factors ranging from 30 to 200 depending on the β_{eff} value. The decrease in the statistical uncertainty induced by this method allows obtaining reliable Monte Carlo estimates for quantities that were notoriously difficult to properly converge. A generalization to the case of family-wise β^j_{eff} was proposed and successfully tested.

An interesting extension of the variance-reduction method proposed in Sec. 3 would be to generate delayed neutrons for each precursor family j in order to obtain accurate sensitivities for the effective delayed neutron fraction β^j_{eff} per precursor family.

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