Electromagnetic and gravitational local spatial densities for hadrons

Julia Panteleeva[1,*]

[1]Institute for Theoretical Physics II, Ruhr University Bochum, D-44780 Bochum, Germany

Abstract. The novel definition of electromagnetic and gravitational local spatial densities of hadrons in zero average momentum frame are considered. The connection of these densities with the densities in infinite-momentum frame and the comparison with densities in the static approximation (Breit-frame densities) are discussed.

1 Introduction

The form factors measured in experiments provide values for such fundamental characteristics as charge, mass etc. The charge radius, which attracts much attention recently, appears when one defines the charge density, otherwise one has just a term proportional to the derivative of the form factor at zero momentum transfer appearing in the expansion of the form factor, without any interpretation. Since the form factors describe the deviation from the point-like particle in the momentum space, they can be connected with the spatial densities, although it might not be adequate for describing the system from microcosm. One could consider this connection as an attempt to extract more information from the form factors and to better understand the spatial structure of particles.

There are different ways to define the spatial densities. The traditional definition of spatial densities as three-dimensional Fourier transforms of the corresponding form factors in Breit-frame (here it is identified as the static approximation) was first defined for the electromagnetic case in Refs. [1–3] and then generalized to the gravitational form factors in Refs. [4, 5]. As it has been repeatedly pointed out, this definition is not fully relativistic and can be used only for systems whose Compton wavelengths are much smaller than the charge radius, see e.g. Refs. [6–9]. The fully relativistic densities can be defined in infinite momentum frame (IMF), however, one obtains only two-dimensional distributions which are less intuitive. Alternatively, the phase-space approach allows one to define fully relativistic, three-dimensional densities. However, these densities do not possess a strict probabilistic interpretation, see Ref [10].

In the following the new definition for the three-dimensional fully relativistic spatial densities in zero average momentum frame (ZAMF) is considered. These densities have probabilistic interpretation and are valid for any system independently of its Compton wavelength. The definition is based on the construction of the matrix element of a local operator sandwiched between two wave packet states, which are sharply localized in space. Similar definition for spin-0 system was suggested long ago in overlooked Ref. [11]. Comments on the definition can be also found in Ref. [12].

*e-mail: Juliia.Panteleeva@rub.de

2 A definition of the electromagnetic spatial densities in ZAMF

We start with considering the electromagnetic densities of a spin-1/2 system in its ZAMF, that was defined in Ref. [13]. The electromagnetic densities for the spin-0, -1 and -3/2 systems and also generalization to the gravitational case can be found in Refs. [14–17]. We consider the electric charge operator $\hat{Q}$ with the eigenvalue Q so that $\hat{Q}|p\rangle = Q|p\rangle$, where the momentum eigenstates are normalised as follows $\langle p', s'|p, s\rangle = 2E(2\pi)^3\delta_{s's}\delta^{(3)}(\mathbf{p}' - \mathbf{p})$ with spins s, s', momentum $p = (E, \mathbf{p})$ and energy $E = \sqrt{m^2 + \mathbf{p}^2}$. Suppose, the electromagnetic density operator $\hat{j}^\mu(\mathbf{x}, 0)$ for $t = 0$ can be connected with the charge operator as following

$$\hat{Q} = \int d\mathbf{x}\, \hat{j}^\mu(\mathbf{x}, 0), \tag{1}$$

then the matrix element of the electromagnetic density operator sandwiched between two one-particle eigenstates of the energy-momentum operator should automatically define the electric charge density, which is usually parametrized in terms of the form factors $F_1(q^2)$ and $F_2(q^2)$:

$$\langle p', s'|\hat{j}^\mu(\mathbf{x}, 0)|p, s\rangle = e^{-i(\mathbf{p}'-\mathbf{p})\cdot\mathbf{x}}\, \bar{u}(p', s')\left[\gamma^\mu F_1(q^2) + \frac{1}{2} i\sigma^{\mu\nu} q_\nu\, F_2(q^2)\right] u(p, s), \tag{2}$$

with the normalisation of the Dirac spinors as $\bar{u}(p, s')u(p, s) = 2m\delta_{s's}$, the momentum transfer is given by $q = p' - p$, the form factors are normalized as $F_1(0) = 1$ and $F_2(0) = \kappa/m$, with κ being the anomalous magnetic moment of the spin-1/2 particle.

For the definition of the spatial density one needs a coordinate, which is defined as a relative quantity. The coordinate $\mathbf{x}$ in Eqs. (1) and (2) is not such a quantity. Following the logic of Ref. [9] we define the electromagnetic density as a matrix element of the charge operator between two states, which are localised in coordinate space

$$j^\mu(\mathbf{r}) = \langle \Phi, \mathbf{X}, s'|\hat{j}^\mu(\mathbf{x}, 0)|\Phi, \mathbf{X}, s\rangle. \tag{3}$$

Here a normalizable Heisenberg-picture state is constructed as the following wave packet state in ZAMF – the Lorentz frame in which the expectation value of the three-momentum is zero:

$$|\Phi, \mathbf{X}, s\rangle = \int \frac{d^3p}{\sqrt{2E(2\pi)^3}}\, \phi(s, \mathbf{p})\, e^{-i\mathbf{p}\cdot\mathbf{X}}|p, s\rangle, \tag{4}$$

with the profile function $\phi(s, \mathbf{p})$ and the position of the center-of-charge of the system $\mathbf{X}$, so that $\mathbf{r} = \mathbf{x} - \mathbf{X}$. Since the space where the system is localized is rotationally invariant we use spherically symmetric profile function of the following form

$$\phi(\mathbf{p}) = R^{3/2}\, \tilde{\phi}(R\mathbf{p}), \tag{5}$$

where R characterizes the size of the wave packet such that sharp localization corresponding to small values of R describes the absolutely localized system.

The new densities in the sharp localization approach appear by computing the matrix element of Eq. (3) using the method of dimensional counting developed in Ref. [18] by taking $R \to 0$ without any approximation for the form factors except the requirement that $F_1(q^2)$ and $F_2(q^2)$ decrease at large q^2 faster than $1/q^2$ and $1/q^4$, respectively. The resulting densities are

$$\begin{aligned} J^0(\mathbf{r}) &= \int \frac{d^3q}{(2\pi)^3}\, e^{-i\mathbf{q}\cdot\mathbf{r}} \int_{-1}^{+1} d\alpha \frac{1}{2} F_1\left[(\alpha^2 - 1)\mathbf{q}^2\right], \\ \mathbf{J}(\mathbf{r}) &= \frac{\boldsymbol{\nabla}_\mathbf{r} \times \boldsymbol{\sigma}}{2m} \int \frac{d^3q}{(2\pi)^3}\, e^{-i\mathbf{q}\cdot\mathbf{r}} \int_{-1}^{+1} d\alpha\, \frac{1}{4}\left(1 + \alpha^2\right) m F_2\left[(\alpha^2 - 1)\, \mathbf{q}^2\right]. \end{aligned} \tag{6}$$

We use J^μ instead of j^μ to indicate that these densities are written as operators in spin space.

For better understanding of these densities one can rewrite them using the unit vector $\hat{\mathbf{n}} = \frac{\mathbf{n}}{|\mathbf{n}|}$ in the following form

$$J^0(\mathbf{r}) = \frac{1}{4\pi}\int d\hat{\mathbf{n}}\,\rho_{1,\hat{\mathbf{n}}}(\mathbf{r}), \qquad \mathbf{J}(\mathbf{r}) = \frac{1}{4\pi}\int d\hat{\mathbf{n}}\,\frac{1}{2m}\boldsymbol{\nabla}_{\mathbf{r}}\times\boldsymbol{\sigma}_\perp\,\rho_{2,\hat{\mathbf{n}}}(\mathbf{r}), \tag{7}$$

where

$$\rho_{i,\hat{\mathbf{n}}}(\mathbf{r}) = \rho_i(r_\perp)\,\delta(r_\parallel)\,, \tag{8}$$

and the two-dimensional auxiliary densities $\rho_i(r_\perp)$ are defined in terms of the form factors $F_i(q^2)$ via

$$\rho_1(r_\perp) = \int \frac{d^2q_\perp}{(2\pi)^2}\,e^{-i\mathbf{q}_\perp\cdot\mathbf{r}_\perp}\,F_1\left(-\mathbf{q}_\perp^2\right), \qquad \rho_2(r_\perp) = \int \frac{d^2q_\perp}{(2\pi)^2}\,e^{-i\mathbf{q}_\perp\cdot\mathbf{r}_\perp}\,mF_2\left(-\mathbf{q}_\perp^2\right). \tag{9}$$

Here and in what follows, $\mathbf{a}_\parallel \equiv \mathbf{a}\cdot\hat{\mathbf{n}}\,\hat{\mathbf{n}}$ ($\mathbf{a}_\perp \equiv \mathbf{a} - \mathbf{a}\cdot\hat{\mathbf{n}}\,\hat{\mathbf{n}} = \hat{\mathbf{n}}\times(\mathbf{a}\times\hat{\mathbf{n}})$) denote the component of a vector $\mathbf{a}$ parallel (perpendicular) to the vector $\mathbf{n}$, and $a_\parallel \equiv |\mathbf{a}_\parallel|$, $a_\perp \equiv |\mathbf{a}_\perp|$.

It is worth to mention two points here. First of all, the obtained densities are independent of the Compton wavelength of the system and of the details of the wave packet states in which the system was prepared. Second, the two-dimensional auxiliary densities in Eq. (9) describe the densities in IMF, see Refs. [6, 7]. Comparing the densities in both frames, it becomes clear that the three-dimensional ZAMF densities in Eq. (7) can be reconstructed from the two-dimensional IMF densities via averaging the IMF densities in all possible directions. However, as it was obtained in Ref. [16] this kind of "holographic" connection between ZAMF and IMF densities is functioning only for systems with symmetric spin structure, e.g. the connection of ZAMF and IMF densities for the quadrupole structure is more complicated. For the detailed calculation and discussions of the densities in moving frames in the current approach see Refs. [12–14, 16, 17].

3 Definition of gravitational spatial densities in ZAMF

Although a superposition of one-particle eigenstates of the four-momentum operator with different four-momenta is not an eigenstate of the same operator as it was in the case of the charge operator, such a state does not contain admixtures of states with particle-antiparticle pairs and, therefore, the wave packet state can be used to compute the matrix element of the energy-momentum tensor (EMT) operator. Analogically to the electromagnetic density we define spatial gravitational densities using the following matrix element of a local EMT operator between two wave packet states

$$t^{\mu\nu}(\mathbf{r}) = \langle\Phi,\mathbf{X},s'|\hat{T}^{\mu\nu}(\mathbf{x},0)|\Phi,\mathbf{X},s\rangle, \tag{10}$$

where the matrix element of the EMT operator between two one-particle energy-momentum eigenstates is parametrised in terms of the gravitational form factors as follows

$$\langle p',s'|\hat{T}^{\mu\nu}(\mathbf{x},0)|p,s\rangle = e^{-i\mathbf{q}\cdot\mathbf{x}}\bar{u}(p',s')\left[A(q^2)\frac{P^\mu P^\nu}{m} + iJ(q^2)\frac{P^\mu\sigma^{\nu\alpha}q_\alpha + P^\nu\sigma^{\mu\alpha}q_\alpha}{2m}\right.$$
$$\left. + D(q^2)\frac{q^\mu q^\nu - g^{\mu\nu}q^2}{4m}\right]u(p,s)\,. \tag{11}$$

For the form factors $A(q^2)$, $J(q^2)$ and $D(q^2)$ decreasing for large q^2 as $1/q^2$, $1/(q^2)^{3/2}$ and $1/q^4$ or faster, respectively, we obtain the following spatial EMT densities in spin space

$$
\begin{aligned}
t^{00}(\mathbf{r}) &= N_{\phi,R} \int \frac{d^2\hat{n}\, d^3q}{(2\pi)^3} A\left[-\mathbf{q}_\perp^2\right] e^{-i\mathbf{q}\cdot\mathbf{r}}, \\
t^{0i}(\mathbf{r}) &= N_{\phi,R} \frac{i\varepsilon^{jkl}}{2m} \int \frac{d^2\hat{n}\, d^3q}{(2\pi)^3} \sigma_\perp^k q^l \left(\delta^{ij} + \hat{n}^i \hat{n}^j\right) J\left[-\mathbf{q}_\perp^2\right] e^{-i\mathbf{q}\cdot\mathbf{r}}, \\
t_2^{ij}(\mathbf{r}) &= \frac{1}{2} N_{\phi,R,2} \int \frac{d^2\hat{n}\, d^3q}{(2\pi)^3} \left(q^i q^j - \delta^{ij}\mathbf{q}_\perp^2\right) D\left(-\mathbf{q}_\perp^2\right) e^{-i\mathbf{q}\cdot\mathbf{r}},
\end{aligned} \tag{12}
$$

with $\mathbf{r}$ being the position of the center-of-gravity of the system and normalization constants:

$$
N_{\phi,R} = \frac{1}{R} \int d\tilde{P} \tilde{P}^3 |\tilde{\phi}(|\tilde{\mathbf{P}}|)|^2, \qquad N_{\phi,R,2} = \frac{R}{2} \int d\tilde{P} \tilde{P} |\tilde{\phi}(|\tilde{\mathbf{P}}|)|^2. \tag{13}
$$

Notice, that we decomposed the ij-component of the EMT into two parts, namely, the flow and stress tensors and showed here only the stress tensor t_2^{ij} characterising the internal structure of the system, for details see Ref. [17].

In contrast to the electromagnetic densities, the dependence on the information about the wave packet states, in which the system was prepared, remains in gravitational densities, however, only as an overall normalisation of densities. The normalization factor $N_{\phi,R}$ for the energy (t^{00}) and the momentum distributions (t^{0i}) rises strongly in the sharp localization limit $R \to 0$, which is expected because the absolute localisation of the system requires a huge amount of energy. The vanishing of the normalisation factor $N_{\phi,R,2}$ in front of the stress tensor is also expected because this part of the EMT is related to the variation of the action with respect to the spatial part of the metric $g_{ij}(r)$. This variation corresponds to a change of a system's location in three-dimensional space, which vanishes for sharply localized states.

4 Comparison of ZAMF with static approximation

Starting from the definition of the electromagnetic densities in ZAMF in Eq. (3) we consider the static approximation of these densities following the logic of Ref. [9]. For this we compute the matrix element of Eq. (3) by first expanding in powers of $1/m$ and then taking the sharp localization limit $R \to 0$. Using the method of dimensional counting of Ref. [18] we obtain the traditional electromagnetic densities in static approximation known as densities in Breit-frame and obtained long ago by Sachs in Ref. [3]

$$
\begin{aligned}
J^0_{\text{static}}(\mathbf{r}) &= \int \frac{d^3q}{(2\pi)^3} e^{-i\mathbf{q}\cdot\mathbf{r}} \left(F_1(-\mathbf{q}^2) + \frac{q^2}{4m} F_2(-\mathbf{q}^2)\right), \\
\mathbf{J}_{\text{static}}(\mathbf{r}) &= \frac{\boldsymbol{\nabla}_\mathbf{r} \times \boldsymbol{\sigma}}{2m} \int \frac{d^3q}{(2\pi)^3} e^{-i\mathbf{q}\cdot\mathbf{r}} \left(F_1(-\mathbf{q}^2) + m F_2(-\mathbf{q}^2)\right).
\end{aligned} \tag{14}
$$

Using the same procedure for the gravitational densities defined in Eq. (10) we obtain the static approximation of gravitational spatial densities [5]:

$$
t^{00}_{\text{static}}(\mathbf{r}) = m \int \frac{d^3q}{(2\pi)^3} A\left(-\mathbf{q}^2\right) e^{-i\mathbf{q}\cdot\mathbf{r}}, \quad t^{0i}_{\text{static}}(\mathbf{r}) = -\frac{i}{2}\epsilon^{ijk}\sigma^k \int \frac{d^3q}{(2\pi)^3} q^j J\left(-\mathbf{q}^2\right) e^{-i\mathbf{q}\cdot\mathbf{r}},
$$
$$
t^{ij}_{2,\text{static}}(\mathbf{r}) = \frac{1}{4m} \int \frac{d^3q}{(2\pi)^3} D\left(-\mathbf{q}^2\right) \left(-\mathbf{q}^2\delta^{ij} + q^i q^j\right) e^{-i\mathbf{q}\cdot\mathbf{r}}. \tag{15}
$$

The resulting densities in static approximation correspond to the spatial densities for packets localized with R much bigger than the Compton wavelength $1/m$ while much smaller than any other length scale characterizing the system. Such definition, however, becomes improper for systems like light hadrons, whose characteristic length scales are comparable to the Compton wavelength or smaller, see Ref. [9].

It is obvious that the densities in the static approximation differ from the densities in the sharp localisation approach. In the following it is shown how this difference results in various radii of the system. The mean square charge radius of the system (the second moment of the charge distribution) is related to the form factor slope in static approximation and in sharp localization approach via

$$\langle r^2\rangle^C_{\rm static} \simeq 6\left(\frac{dF_1(0)}{dq^2}+\frac{F_2(0)}{4m}\right) \quad \text{and} \quad \langle r^2\rangle^C \simeq 4\frac{dF_1(0)}{dq^2}. \tag{16}$$

The size of the proton measured by the new charge distribution is $\sqrt{\langle r_{\rm p}^2\rangle} \simeq 0.63$ fm, which differs from the static approximation result $\sqrt{\langle r_{\rm p}^2\rangle_{\rm static}} \simeq 0.84$ fm, see Ref. [19]. This discrepancy, however, has no practical implications since the radius extracted from the electron-proton scattering as well as from the electronic and muonic hydrogen atoms is defined based on the expansion of the electric Sachs form factor around $q^2 = 0$. The radius defined via sharply localized packets is smaller due to the squeezing of the density as explained in Ref. [14]. This also applies to the magnetic radius.

The mean square energy radius related to the energy distribution and the mean square mechanical radius related to the normal force distribution of the stress tensor in static approximation [20], and in the sharp localisation approach[1] have the following forms

$$\langle r^2\rangle^E_{\rm static} \simeq 6\frac{dA(0)}{dq^2} \quad \text{and} \quad \langle r^2\rangle^{\rm mech}_{\rm static} \simeq \frac{6D(0)}{\int_{-\infty}^{0} dt D(t)},$$
$$\langle r^2\rangle^E \simeq 4\frac{dA(0)}{dq^2} \quad \text{and} \quad \langle r^2\rangle^{\rm mech} \simeq \frac{6D(0)}{\int_{-\infty}^{0} dt \int_{-1}^{1} \frac{d\alpha}{2} D[t(1-\alpha^2)]}. \tag{17}$$

The same smaller mean squared charge and energy radii, compared to the Breit-frame results, have been obtained previously in Refs. [8, 21] using light-front coordinates. From the physical point of view the squeezing of densities and size of the system in the sharp localisation approach is expected, because the system is absolutely localized so that in some imaginary direct experiment on size of some particle, what would be measured is only the size of the system without any delocalisation effects in contrast to the static approximation, where also delocalization effects contributing to the size would be measured.

5 Applications and Conclusion

For studying of three-dimensional spatial structure of hadrons it is important to have a suitable definition of local densities. The suggested definition of spatial densities in sharp localisation approach in terms of experimentally measured form factors can be used for any system independently of its Compton wavelength in contrast to the densities in the static approximation, where the densities are valid only for systems whose intrinsic size is larger or comparable

[1]The definition of mechanical radius in the sharp localiazion approach differs from the corresponding definition in the static approximation of Ref. [20] because the normal force distribution is modified due to additional function coming from the time-dependent part of the conservation of the EMT, details can be found in Ref. [22].

with its Compton wavelength. The suggested densities are fully relativistic, i.e. they do not need any relativistic corrections. Moreover, the "holographic" connection of ZAMF and IMF densities can be traced, that might give interesting insights into the understanding of spatial structure of hadrons.

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