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A perturbation method for the ray-transfer matrix of the crystalline lens

Antonio Barion[1,*] and *Koondanibha* Mitra[1]

[1]Eindhoven University of Technology, PO Box 513, 5600 MB Eindhoven, The Netherlands

Abstract. Analytical expressions for the ray-transfer matrix have been proven useful for the understanding of ray propagation in gradient-index (GRIN) lenses. The determination of an exact analytical expression for the ray-transfer matrix of arbitrary GRIN lenses remains unsolved. We propose an approximation based on the perturbation method with highly accurate results for models of the crystalline lens, which outperforms existing methods.

1 Introduction

Understanding the paraxial propagation of rays in optical systems is crucial for their characterization, e.g., for determining their power, cardinal planes, etc. The paraxial propagation is often described by a ray-transfer matrix transferring the source coordinates to target coordinates.

Providing analytical expressions for the ray-transfer matrix of an arbitrary gradient-index (GRIN) lens is not possible. However, as has been investigated in the comparative study [1], approximate expressions can be derived.

The aim of this work is to determine more accurate analytical expressions for the ray-transfer matrix. Using a perturbation method for the Lagrangian equation of paraxial optics, we present an approach that outperforms previous methods, e.g., the parabolic method in [1], when applied to models for the crystalline lens.

2 Lagrangian Optics

In our model we consider GRIN lenses that are rotationally symmetric around the optical axis. Let $(\boldsymbol{q}, z)$ be the coordinates in space, where $\boldsymbol{q} = (q_x, q_y)$. Due to symmetry, the refractive index $n(\boldsymbol{q}, z)$ is modeled by

$$n(\boldsymbol{q}, z) = n_0(z) + n_2(z)|\boldsymbol{q}|^2 + n_4(z)|\boldsymbol{q}|^4 + \cdots . \tag{1}$$

Let $\dot{\boldsymbol{q}} = \mathrm{d}\boldsymbol{q}/\mathrm{d}z$, then the paraxial equation of ray propagation reads [2]

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[n_0(z)\dot{\boldsymbol{q}}\right] = 2n_2(z)\boldsymbol{q}. \tag{2}$$

We introduce a reparametrization of the optical axis coordinate according to

$$\eta(z) = \int_0^z \frac{1}{n_0(z')}\mathrm{d}z'. \tag{3}$$

With a some abuse of notation, the new paraxial equation reads

$$\frac{\mathrm{d}^2\boldsymbol{q}}{\mathrm{d}\eta^2} = 2n_0(\eta)n_2(\eta)\boldsymbol{q}. \tag{4}$$

Eq. (4) can be solved exactly if

$$2n_0(\eta)n_2(\eta) = k, \quad k \in \mathbb{R}, \tag{5}$$

and reduces to a harmonic oscillator equation for $k < 0$. The resulting paraxial ray-transfer matrix $\mathbf{M}(z)$ is given by [3]

$$\begin{pmatrix} \boldsymbol{q}(z) \\ n_0(z)\dot{\boldsymbol{q}}(z) \end{pmatrix} = \underbrace{\begin{pmatrix} \cosh(\varphi(z)) & \frac{1}{\sqrt{k}}\sinh(\varphi(z)) \\ \sqrt{k}\sinh(\varphi(z)) & \cosh(\varphi(z)) \end{pmatrix}}_{\mathbf{M}(z)} \begin{pmatrix} \boldsymbol{q}(0) \\ n_0(0)\dot{\boldsymbol{q}}(0) \end{pmatrix}, \tag{6a}$$

$$\varphi(z) = \sqrt{k}\int_0^z \frac{1}{n_0(z')}\mathrm{d}z'. \tag{6b}$$

It holds that $\det\left(\mathbf{M}(z)\right) = 1\ \forall z$, which implies conservation of étendue. The cylindrical GRIN lens with n_0 and n_2 constant and the planar GRIN lens with $n_2(z) = 0$ both satisfy condition (5). As such, the matrix in Eq. (6) is a generalization of the these two known cases [3].

However, in general the condition $2n_0(\eta)n_2(\eta) = k$ does not hold and an exact expression like in Eq. (6) cannot be obtained. In the comparative study performed in [1] the best candidate for obtaining an approximate expression is the so-called parabolic method. The parabolic method assumes a parabolic trajectory of the ray inside the GRIN lens. Below, we propose an alternative.

3 The Perturbation Method

Consider the equation

$$\frac{\mathrm{d}^2\boldsymbol{q}}{\mathrm{d}\eta^2} = f(\eta)\boldsymbol{q}, \tag{7}$$

where $f(\eta) = 2n_0(\eta)n_2(\eta)$. Let us assume that

$$f(\eta) = k + \varepsilon P(\eta), \tag{8}$$

where $|\varepsilon| \ll 1$ is the perturbation parameter and $P(\eta)$ is a suitable function. The parameter ε is proportional to the maximum deviation of $2n_0(\eta)n_2(\eta)$ from the value k.

*e-mail: a.barion@tue.nl

The solution $\boldsymbol{q}(\eta)$ is then expanded according to:

$$\boldsymbol{q}(\eta) = \boldsymbol{q}_0(\eta) + \varepsilon \boldsymbol{q}_1(\eta) + \varepsilon^2 \boldsymbol{q}_2(\eta) + \cdots . \quad (9)$$

We substitute Eq. (9) into Eq. (7) and equate terms of equal powers in ε. The solution $\boldsymbol{q}_0(\eta)$ to the zeroth-order equation is as in Eq. (6). The first-order solution $\boldsymbol{q}_1(\eta)$ is solved using the method of variation of parameters involving $P(\eta)$.

After reversing the reparametrization to the z - coordinate, it can be shown that

$$\|\boldsymbol{q}(z) - (\boldsymbol{q}_0(z) + \varepsilon \boldsymbol{q}_1(z))\|_2 = O(\varepsilon^2). \quad (10)$$

The approximate solution can now be written as

$$\begin{pmatrix} \boldsymbol{q}_0(z) + \varepsilon \boldsymbol{q}_1(z) \\ n_0(z)(\dot{\boldsymbol{q}}_0(z) + \varepsilon \dot{\boldsymbol{q}}_1(z)) \end{pmatrix} = [\mathbf{M}_0(z) + \varepsilon \mathbf{M}_1(z)] \begin{pmatrix} \boldsymbol{q}(0) \\ n_0(0)\dot{\boldsymbol{q}}(0) \end{pmatrix}. \quad (11)$$

Here, $\det\big(\mathbf{M}_0(z) + \varepsilon \mathbf{M}_1(z)\big) \neq 1$, but it can be corrected by adding a term of order ε^2 to the matrix. This correction ensures again conservation of étendue and does not degrade the quality of the result as it is of equal order as the approximating error; see Eq. (10).

4 Numerical Results

We compare our results to the parabolic method described in [1]. Two models of the crystalline lens are considered: the Gullstrand and the Huang & Moore model [4, 5]; see Tables 1 and 2. In our work we choose $P(\eta)$ in Eq. (8) to be a quadratic polynomial and fit it to the product $2n_0(\eta)n_2(\eta)$. The perturbation parameter for the Gullstrand model is $\varepsilon_G \approx 10^{-3}$ and for the Huang & Moore model $\varepsilon_{HM} \approx 10^{-4}$.

$n_0(z)$	$1.386 + 2.4637 \times 10^{-2} z$ $-8.22384 \times 10^{-3} z^2 + 3.834 \times 10^{-4} z^3$
$n_2(z)$	$-1.2318 \times 10^{-3} + 8.904 \times 10^{-4} z$ $-2.7875 \times 10^{-4} z^2$

Table 1. GRIN coefficients of the Gullstrand model [1, 4].

$n_0(z)$	$1.3678 + 3.455 \times 10^{-2} z + 1.103 \times 10^{-3} z^2$ $-1.5657 \times 10^{-2} z^3 + 6.855 \times 10^{-3} z^4$ $-1.065 \times 10^{-3} z^5 + 9.9 \times 10^{-6} z^7$ $-1.938 \times 10^{-3} z^{2/5} + 9.78 \times 10^{-3} z^{2/3}$
$n_2(z)$	-1.978×10^{-3}

Table 2. GRIN coefficients of the Huang and Moore model [1, 5].

The reference solution is taken to be the numerical solution to the matrix ODE associated to Eq. (2) using a second-order Magnus integrator with an accuracy of 10^{-10} [3]. We denote by $\mathrm{err}_{\mathrm{parab}}$ the 2-norm of the difference between the matrix of the parabolic method and the reference solution. Similarly, for $\mathrm{err}_{\mathrm{pert}}$ we compare our proposed matrix to the reference matrix.

The performance of the two methods can be seen in Table 3. The error of the matrix resulting from the perturbation method is of the order ε^2.

	$\mathrm{err}_{\mathrm{parab}}$	$\mathrm{err}_{\mathrm{pert}}$
Gullstrand	2.26×10^{-4}	1.64×10^{-6}
Huang & Moore	1.82×10^{-2}	5.24×10^{-8}

Table 3. 2-norm matrix errors of the two analytical approximation methods.

5 Conclusion

It is not possible to determine an analytical expression for the ray-tracing matrix of an arbitrary GRIN lens. We provide an analytical approximation to it which generalizes the already known matrices for cylindrical and planar GRINs.

The presented perturbation method allows us to determine highly accurate formulas. The assumptions made on the parameters of the refractive index are not particularly stringent. Furthermore, these assumptions appear to be generally satisfied by the crystalline lens of the eye as we apply the proposed method to two different eye models with satisfactory results.

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