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Local alpha strength functions for alpha knockout reactions

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Abstract. We have recently proposed "local α-removal strength function", which quantifies the α-particle distribution inside the nucleus, using the mean-field-type wave function. Some updates, including the finite-size effect and the α knockout amplitude, are presented.

1 Introduction

The α clustering is a characteristic feature of nuclear structure, observed in both light and heavy nuclei. Among many microscopic studies, the antisymmetric molecular dynamics (AMD) [2] and the fermionic molecular dynamics (FMD) [1] were extensively utilized in studies of the nuclear cluster phenomena and heavy-ion reactions. In the AMD ad FMD, the gaussian wave packet is assumed for a single-particle state [3, 4]. Each gaussian wave packet has parameters corresponding to the central position and the width. Thus, the α-like correlation is identified by close location of centers of four gaussian wave packets.

In contrast, the energy density functional (EDF) theory, which is capable of describing light and heavy nuclei using a single energy density functional, can provide the optimal single-particle wave functions to minimize the total energy of a Slater determinant. The pairing correlation can also be treated in the BCS or in the Bogoliubov theory [5]. However, it is not easy to find α correlation in terms of the mean-field-type wave functions, because each single-particle wave function is spread over the nucleus. Although there are some attempts to find clustering in the mean-field method [6, 7], it is not straightforward to study heavy systems.

We have recently proposed the local α-removal strength function which is easily calculable in the EDF approaches. It quantifies the probability and the distribution of the α particles inside the nucleus, using the mean-field-type wave functions. It tells us not only the transition to the ground state, but also transitions to excited states in the residual nucleus. We have performed a numerical computation for even-even Sn nuclei. The local α-removal strength function provides a simple picture consistent with a recent experiment measurement of the quasi-free α-knockout reactions [8].

In the present paper, we propose a possible improvement of the calculation of the local α-removal strength functions. In addition, the α knockout amplitude will be discussed.

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2 Local α-removal strength functions

In this section, we briefly recapitulate the local α-removal strength functions of Ref. [9]. First of all, the α-removal operator at a position $\mathbf{r}$ is defined as

$$\hat{\alpha}(\mathbf{r}) \equiv \prod_{k=1}^{4} \int d\mathbf{r}_k \phi_\alpha^{\mathbf{r}*}(\mathbf{r}_k) \hat{\psi}_{\sigma_k}^{(q_k)}(\mathbf{r}_k) \quad (1)$$

where the spin-isospin indexes are

$$\sigma_k = \begin{cases} \uparrow & \text{for } k = 1, 3 \\ \downarrow & \text{for } k = 2, 4 \end{cases}, \qquad q_k = \begin{cases} n & \text{for } k = 1, 2 \\ p & \text{for } k = 3, 4 \end{cases}. \quad (2)$$

$\hat{\psi}_\sigma^{(q)}(\mathbf{r})$ indicates the field operator for the particle with the isospin $q = n, p$ and the spin $\sigma = \uparrow, \downarrow$. The wave function $\phi_\alpha^{\mathbf{r}}(\mathbf{r}')$ is typically a Gaussian form,

$$\phi_\alpha^{\mathbf{r}}(\mathbf{r}') = \left(\frac{\nu}{\pi}\right)^{3/4} e^{-\nu|\mathbf{r}'-\mathbf{r}|^2/2}. \quad (3)$$

In Eq. (1), the exact center-of-mass coordinate of α particle is given by $\sum_k \mathbf{r}_k/4$, while $\mathbf{r}$ can be regarded as its approximate representation.

The local α-removal strength function is given as

$$S_\alpha(\mathbf{r}, E) \equiv \langle\Phi_0|\, \hat{\alpha}^\dagger(\mathbf{r})\delta(E - \hat{H})\hat{\alpha}(\mathbf{r})\, |\Phi_0\rangle = \sum_{n=0}^{\infty} \left|\langle\Phi_n'|\, \hat{\alpha}(\mathbf{r})\, |\Phi_0\rangle\right|^2 \delta(E - E_n^{A-4}) \quad (4)$$

where $\hat{H}$ is the Hamiltonian, and $|\Phi_0\rangle$ is the ground state of the nucleus (N, Z) while $|\Phi_n'\rangle$ is the ground $(n = 0)$ and excited states of the residual nucleus $(N - 2, Z - 2)$.

In practice, we use the Hartree-Fock-plus-BCS theory to approximate the ground and excited states of $|\Phi_0\rangle$ and $|\Phi_n'\rangle$. In addition, we neglect the rearrangement effect of the mean fields, assuming that the removal of an α particle does not change the mean-field potential and the superfluid (BCS) states. We also assume that states $|\Phi_0\rangle$ and $|\Phi_n'\rangle$ are given by product wave functions of protons and neutrons. With these approximation, $S_\alpha(\mathbf{r}; E)$ of Eq. (4) can be expressed in a product form of proton and neutron strengths.

$$S_\alpha(\mathbf{r}, E) = \sum_{k=0}^{\infty} \sum_{k'=0}^{\infty} F_k^{(n)}(\mathbf{r}) F_{k'}^{(p)}(\mathbf{r}) \delta(E - E_{kk'}), \quad (5)$$

where k (k') stands for neutron (proton) 0qp and 2qp excitations. $F_k^{(q)}(\mathbf{r})$ is defined by

$$F_k^{(q)}(\mathbf{r}) \equiv \left| \left\langle \Phi_k' \middle| \int d\mathbf{r}_1 \phi_\alpha^{\mathbf{r}*}(\mathbf{r}_2) \hat{\psi}_\downarrow^{(n)}(\mathbf{r}_2) \int d\mathbf{r}_2 \phi_\alpha^{\mathbf{r}*}(\mathbf{r}_1) \hat{\psi}_\uparrow^{(n)}(\mathbf{r}_1) \middle| \Phi_0 \right\rangle \right|^2, \quad (6)$$

for $q = n$ and p. Furthermore, in Ref. [9], we adopt a point-α approximation, namely, replacing $\phi_\alpha^{\mathbf{r}}(\mathbf{r}')$ by the delta function $\delta(\mathbf{r} - \mathbf{r}')$. The calculation of the α-removal strength function $S_\alpha(\mathbf{r}, E)$, is feasible with these approximations [9]. In the next section, we give a possible improvement to lift the restriction of the point-α approximation.

3 Finite-size effect of the α particle

Let us assume that the mean-field Hamiltonian has the time-reversal symmetry. This is usually true at the ground state of even-even nuclei, as $|\Phi_0\rangle$ is invariant with respect to the time-reversal transformation, $\hat{T}|\Phi_0\rangle = |\Phi_0\rangle$ (the same for $|\Phi_0'\rangle$). The quasiparticle operator, $a_i^\dagger$, and its time-reversal partner, $a_{\bar{i}}^\dagger = \hat{T}^\dagger a_i^\dagger \hat{T}$, have the same quasiparticle energies, $E_i = E_{\bar{i}}$.

First, when the state $|\Phi_0\rangle$ is in the superfluid phase ($\Delta \neq 0$), the ground state of the residual nucleus is regarded as $|\Phi_0'\rangle \approx |\Phi_0\rangle$ in the present approximation. We will later discuss the case that $|\Phi_0\rangle$ is in the non-super phase. The ground-ground transition amplitude is given by a product of $F_0^{(n)}(\mathbf{r})$ and $F_0^{(p)}(\mathbf{r})$.

$$F_0(\mathbf{r}) = \left| \int d\mathbf{r}_1 \int d\mathbf{r}_2 \phi_\alpha^*(\mathbf{r}_{1\mathbf{r}}) \phi_\alpha^*(\mathbf{r}_{2\mathbf{r}}) \kappa(\mathbf{r}_1, \mathbf{r}_2) \right|^2, \tag{7}$$

where the pair density is given by

$$\kappa(\mathbf{r}_1, \mathbf{r}_2) \equiv \langle \Phi_0 | \hat{\psi}_\downarrow(\mathbf{r}_2) \hat{\psi}_\uparrow(\mathbf{r}_1) | \Phi_0 \rangle = \sum_i u_i v_i \varphi_i(\mathbf{r}_1, \uparrow) \varphi_{\bar{i}}(\mathbf{r}_2, \downarrow), \tag{8}$$

where φ_i and $\varphi_{\bar{i}}$ are the single-particle (canonical) wave functions in the BCS ground state $|\Phi_0\rangle$. Hereafter, we omit the isospin index (q) for simplicity. We introduce a notation for "orbital overlaps" as

$$\langle \phi_\alpha^\mathbf{r} | \varphi(\sigma) \rangle_o \equiv \int d\mathbf{r}' \phi_\alpha(\mathbf{r}' - \mathbf{r}) \varphi(\mathbf{r},' \sigma), \quad \langle \varphi(\sigma) | \phi_\alpha^\mathbf{r} \rangle_o \equiv \int d\mathbf{r}' \phi_\alpha(\mathbf{r}' - \mathbf{r}) \varphi^*(\mathbf{r}', \sigma). \tag{9}$$

Note that ϕ_α is the real gaussian wave function. Then, $F_0(\mathbf{r})$ is expressed as

$$F_0(\mathbf{r}) = \left| \sum_{i>0} u_i v_i \left(\langle \phi_\alpha^\mathbf{r} | \varphi_i(\uparrow) \rangle_o \langle \phi_\alpha^\mathbf{r} | \varphi_{\bar{i}}(\downarrow) \rangle_o - \langle \phi_\alpha^\mathbf{r} | \varphi_i(\downarrow) \rangle_o \langle \phi_\alpha^\mathbf{r} | \varphi_{\bar{i}}(\uparrow) \rangle_o \right) \right|^2, \tag{10}$$

where $i > 0$ means that the summation is restricted to half of the single-particle states, namely, excluding the time-reversed states $\bar{i}$. Using the time-reversal relations $\varphi_{\bar{i}}(\mathbf{r}, \downarrow) = \varphi_i^*(\mathbf{r}, \uparrow)$ and $\varphi_{\bar{i}}(\mathbf{r}, \uparrow) = -\varphi_i^*(\mathbf{r}, \downarrow)$, we may rewrite Eq. (10) as

$$\begin{aligned} F_0(\mathbf{r}) &= \left| \sum_{i>0} u_i v_i \left(\langle \phi_\alpha^\mathbf{r} | \varphi_i(\uparrow) \rangle_o \langle \varphi_i(\uparrow) | \phi_\alpha^\mathbf{r} \rangle_o + \langle \phi_\alpha^\mathbf{r} | \varphi_i(\downarrow) \rangle_o \langle \varphi_i(\downarrow) | \phi_\alpha^\mathbf{r} \rangle_o \right) \right|^2 \\ &= \left| \sum_{i>0} u_i v_i \left\langle \varphi_i \middle| \hat{P}_\alpha(\mathbf{r}) \middle| \varphi_i \right\rangle \right|^2, \end{aligned} \tag{11}$$

where the projection operator on the single-particle gaussian state centered at the position $\mathbf{r}$ is defined by

$$\hat{P}_\alpha(\mathbf{r}) = \sum_{k=1}^{4} |\phi_\alpha^\mathbf{r}(k)\rangle \langle \phi_\alpha^\mathbf{r}(k)|, \tag{12}$$

where $k = 1, \cdots, 4$ correspond to nucleons with spin and isospin of Eq. (2). These four particles are assumed to have identical orbital wave function.

In the present approximation, excited states are described by two-quasiparticle (2qp) excitations on the ground state $|\Phi_0\rangle$ either in the neutron sector, the proton sector , or the both.

The index k of Eq. (6) corresponds to the two-quasiparticle ij.

$$F_{ij} = v_i^2 v_j^2 \left| \langle \phi_\alpha^{\mathbf{r}} | \varphi_i(\uparrow) \rangle_o \langle \phi_\alpha^{\mathbf{r}} | \varphi_j(\downarrow) \rangle_o - \langle \phi_\alpha^{\mathbf{r}} | \varphi_j(\uparrow) \rangle_o \langle \phi_\alpha^{\mathbf{r}} | \varphi_i(\downarrow) \rangle_o \right|^2 \tag{13}$$

$$= v_i^2 v_j^2 \left| \langle \phi_\alpha^{\mathbf{r}} | \varphi_i(\uparrow) \rangle_o \langle \varphi_{\bar{j}}(\uparrow) | \phi_\alpha^{\mathbf{r}} \rangle_o + \langle \varphi_{\bar{j}}(\downarrow) | \phi_\alpha^{\mathbf{r}} \rangle_o \langle \phi_\alpha^{\mathbf{r}} | \varphi_i(\downarrow) \rangle_o \right|^2 \tag{14}$$

$$= v_i^2 v_j^2 \left| \left\langle \varphi_{\bar{j}} \middle| \hat{P}_\alpha(\mathbf{r}) \middle| \varphi_i \right\rangle \right|^2 . \tag{15}$$

Especially, $F_{i\bar{i}}$ are give by a simple form

$$F_{i\bar{i}} = v_i^4 \left\langle \varphi_i \middle| \hat{P}_\alpha(\mathbf{r}) \middle| \varphi_i \right\rangle^2, \qquad i > 0. \tag{16}$$

Because of the time-reversal symmetry, the 2qp excitations ij, $i\bar{j}$, $\bar{i}j$, and $\bar{i}\bar{j}$ have the same excitation energy. Thus, we sum up these four contributions to define $\tilde{F}_{ij}(\mathbf{r})$ as a partial summation of $F_{ij}(\mathbf{r})$ over the 2qp states with the same energy.

$$\tilde{F}_{ij}(\mathbf{r}) \equiv F_{ij}(\mathbf{r}) + F_{\bar{i}j}(\mathbf{r}) + F_{i\bar{j}}(\mathbf{r}) + F_{\bar{i}\bar{j}}(\mathbf{r}) = v_i^2 v_j^2 \sum_{I=i,\bar{i}} \sum_{L=j,\bar{j}} \left| \left\langle \varphi_J \middle| \hat{P}_\alpha(\mathbf{r}) \middle| \varphi_I \right\rangle \right|^2 \tag{17}$$

$$= 2 v_i^2 v_j^2 \left(\left| \left\langle \varphi_j \middle| \hat{P}_\alpha(\mathbf{r}) \middle| \varphi_i \right\rangle \right|^2 + \left| \left\langle \varphi_{\bar{j}} \middle| \hat{P}_\alpha(\mathbf{r}) \middle| \varphi_i \right\rangle \right|^2 \right) \tag{18}$$

$$= 2 v_i^2 v_j^2 \left\langle \varphi_i \middle| \hat{P}_\alpha(\mathbf{r}) \middle| \varphi_i \right\rangle \left\langle \varphi_j \middle| \hat{P}_\alpha(\mathbf{r}) \middle| \varphi_j \right\rangle, \tag{19}$$

for $i > j > 0$. For convenience, we define $\tilde{F}_{ii} = F_{i\bar{i}}$ $(i > 0)$. The local α-removal strength function is now expressed as

$$\begin{aligned} S_\alpha(\mathbf{r}; E) = {} & F_0^{(n)}(\mathbf{r}) F_0^{(p)}(\mathbf{r}) \delta(E) \\ & + F_0^{(p)}(\mathbf{r}) \sum_{i \geq j > 0} \tilde{F}_{ij}^{(n)}(\mathbf{r}) \delta(E - E_{ij}^{(n)}) + F_0^{(n)}(\mathbf{r}) \sum_{i \geq j > 0} \tilde{F}_{ij}^{(p)}(\mathbf{r}) \delta(E - E_{ij}^{(p)}) \\ & + \sum_{i \geq j > 0} \tilde{F}_{ij}^{(n)}(\mathbf{r}) \sum_{k \geq l > 0} \tilde{F}_{kl}^{(p)}(\mathbf{r}) \delta(E - E_{ij}^{(n)} - E_{kl}^{(p)}), \end{aligned} \tag{20}$$

where we define the origin of energy E as the ground state energy of the residual nucleus.

Finally, we comment on the case for no pairing (normal phase) in the ground state $|\Phi_0\rangle$. In this case, we trivially find $F_0 = 0$. Therefore, in Eq. (20), the first two lines on the right-hand side vanish. The lowest four-quasiparticle-energy states with $v_i = v_j = 1$ (the highest occupied orbitals), which may not be unique, are regarded as the ground state of the residual nucleus $|\Phi_0'\rangle$. A possible spurious degeneracy of the ground state is a result of the crude approximation [9].

In Fig. 1, we show the ground-ground transition strength for ^{112}Sn,

$$S_\alpha^0(\mathbf{r}) = \int_{-\epsilon}^{\epsilon} S_\alpha(\mathbf{r}; E) dE = \left| \left\langle \Phi_0' \middle| \hat{\alpha}(\mathbf{r}) \middle| \Phi_0 \right\rangle \right|^2, \tag{21}$$

where ϵ is a positive infinitesimal. In our previous study of Ref. [9], we adopt the delta-type wave function for $\phi_\alpha^{\mathbf{r}}$ in Eq. (1), which leads to the dimension of $S_\alpha^0(\mathbf{r})$ as L^{-12}. Using the gaussian wave function for $\phi_\alpha^{\mathbf{r}}$, $S_\alpha^0(\mathbf{r})$ becomes dimensionless. Therefore, we cannot compare their absolute magnitude. In Fig. 1, we scale the one to be comparable to the other. The finite-size effect is to shift the location of the peak outward, but the shape is approximately identical.

In summary, to calculate the local α-removal strength function including the finite-size effect of the α particle, all we need is $\left\langle \varphi_i \middle| \hat{P}_\alpha(\mathbf{r}) \middle| \varphi_i \right\rangle$ which can be obtained with the calculation of the overlap functions between single-particle states and a gaussian wave function for the α particle, Eq. (9). This can be easily performed.

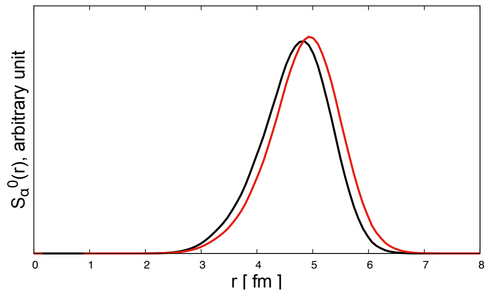

Figure 1. Finite-size effect of the α particle wave function for $S_\alpha^0(r)$. Multiplying a constant $(\nu/\pi)^{3/4}$, it can also be regarded as the α-knockout amplitude $\mathcal{Y}(\mathbf{r})$. (See section 4). The red line corresponds to the calculation of the finite gaussian width. The black line corresponds to the delta-type function, $\phi_\alpha^{\mathbf{R}}(\mathbf{r}) = \delta(\mathbf{r} - \mathbf{R})$. The black line is the same as the one in Ref. [9].

4 α-knockout amplitude

In the cluster model, it is customary to use the α amplitude $\mathcal{Y}(\mathbf{r})$. For decaying states, this is often called the α reduced width amplitude. In the present study, we call it "α-knockout amplitude" because we calculate those for the stable ground state. In this section, we discuss how to define $\mathcal{Y}(\mathbf{r})$ in the EDF approaches. We first rewrite the α-removal operator Eq. (2) in terms of the center-of-mass (CM) coordinate $\mathbf{R}$ and the relative coordinates $\boldsymbol{\xi}_k$ ($k = 1, 2, 3$). Thanks to the simple gaussian form of $|\phi_\alpha^{\mathbf{r}}\rangle$, this can be easily performed with the rearrangement, $\prod_{k=1}^4 \phi_\alpha^{\mathbf{r}}(\mathbf{r}_k) = \Phi_{\rm CM}^{\mathbf{r}}(\mathbf{R})\phi_{\rm rel}(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3)$.

$$\hat{\alpha}(\mathbf{r}) = \int d\mathbf{R} d\boldsymbol{\xi}_1 d\boldsymbol{\xi}_2 d\boldsymbol{\xi}_3 \Phi_{\rm CM}^{\mathbf{r}*}(\mathbf{R}) \phi_{\rm rel}^*(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3) \prod_{k=1}^4 \hat{\psi}_{\sigma_k}^{(q_k)}(\mathbf{r}_k) = \int d\mathbf{R} \Phi_{\rm CM}^{\mathbf{r}*}(\mathbf{R}) \hat{\alpha}^{\mathbf{R}}, \tag{22}$$

where the CM wave function is given by

$$\Phi_{\rm CM}^{\mathbf{r}}(\mathbf{R}) = \left(\frac{4\nu}{\pi}\right)^3 e^{-2\nu|\mathbf{R}-\mathbf{r}|^2}. \tag{23}$$

The α-knockout operator $\hat{\alpha}^{\mathbf{R}}$ is defined in the same manner as $\hat{\alpha}(\mathbf{r})$ replacing $\Phi_{\rm CM}^{\mathbf{r}}(\mathbf{R})$ by $\delta(\mathbf{R} - \sum_{k=1}^4 \mathbf{r}_k/4)$.

$$\hat{\alpha}^{\mathbf{R}} \equiv \int d\mathbf{r}_1 d\mathbf{r}_2 d\mathbf{r}_3 d\mathbf{r}_4 \phi_{\rm rel}^*(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3) \delta\left(\mathbf{R} - \frac{1}{4}\sum_{k=1}^4 \mathbf{r}_k\right) \hat{\psi}_\uparrow^{(n)}(\mathbf{r}_1) \hat{\psi}_\downarrow^{(n)}(\mathbf{r}_2) \hat{\psi}_\uparrow^{(p)}(\mathbf{r}_3) \hat{\psi}_\downarrow^{(p)}(\mathbf{r}_4). \tag{24}$$

Neglecting fluctuation of the CM coordinate in the residual nucleus $A - 4$, the α reduced amplitude is given by[1]

$$\mathcal{Y}(\mathbf{R}) \equiv \left\langle \Phi_0' \middle| \hat{\alpha}^{\mathbf{R}} \middle| \Phi_0 \right\rangle. \tag{25}$$

The present approximations, will reduce this quantity to the calculation of the pair tensor of Eq. (8). However, we have to perform the integration over the relative coordinates $\boldsymbol{\xi}_k$ with the fixed CM coordinate $\mathbf{R}$. To avoid these integrations, we introduce another approximation.

$$\hat{\alpha}(\mathbf{r}) = \int d\mathbf{R} \Phi_{\rm CM}^{\mathbf{r}*}(\mathbf{R}) \hat{\alpha}^{\mathbf{R}} \approx \hat{\alpha}^{\mathbf{r}} \int d\mathbf{R} \Phi_{\rm CM}^{\mathbf{r}*}(\mathbf{R}) = \left(\frac{\pi}{\nu}\right)^{3/4} \hat{\alpha}^{\mathbf{r}}. \tag{26}$$

[1] In literature, it is often denoted as $\mathcal{Y}_L(R)$. It can be obtained by the partial wave expansion of Eq. (25). For instance, see Ref. [10].

The approximation is justified by the fact that the CM gaussian wave function $\Phi^{\mathbf{r}}_{\mathrm{CM}}(\mathbf{R})$ of Eq. (23) is sharply peaked at the position $\mathbf{r}$ and by the assumption that the operator $\hat{\alpha}^{\mathbf{R}}$ is approximately invariant for the region where $\Phi^{\mathbf{r}}_{\mathrm{CM}}(\mathbf{R})$ does not vanish. This approximation leads to a simple conclusion that the α-knockout amplitude is proportional to the α-removal strength to the ground state, $\mathcal{Y}(\mathbf{r}) = \left\langle \Phi'_0 \middle| \hat{\alpha}^{\mathbf{r}} \middle| \Phi_0 \right\rangle = (\nu/\pi)^{3/4} \left\langle \Phi'_0 \middle| \hat{\alpha}(\mathbf{r}) \middle| \Phi_0 \right\rangle$.

5 Summary

We have proposed the local α-removal strength function [9] to quantity the distribution of the α particle inside the nucleus. Using some approximations, such as no rearrangement in the mean fields, it is easy to calculate with the mean-field-type wave functions. In the study of Ref. [9], we adopted the point-α approximation, namely $\phi^{\mathbf{R}}_{\alpha}(\mathbf{r}) = \delta(\mathbf{r} - \mathbf{R})$. In the present paper, we have shown how we can lift this restriction. With the time-reversal properties of the BCS wave functions, necessary quantities can be expressed in elementary forms using the projection operator onto the α-particle gaussian wave function. A result of the numerical calculation for ^{112}Sn seems to indicate that the finite-size effect leads to shifting the peak position of the strength distribution toward the surface, namely, larger r, although the shape of the peak is invariant. A further investigation is required to conclude the effect.

We also show that the α-knockout amplitude can be easily estimated with an assumption that the α-knockout operator $\alpha^{\mathbf{R}}$ is invariant within a region where the CM gaussian wave function $\Phi^{\mathbf{r}}_{\mathrm{CM}}(\mathbf{R})$ is finite. This may open up a practical application of the present quantity for α-knockout reaction theories.

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