On the cluster-shell coexistence

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Abstract. The cluster-shell coexistence is discussed from the viewpoint of the multiconfigurational dynamical symmetry, which is the common intersection of the shell, collective and cluster models for the multi major-shell problem.

1 Introduction

The basic difficulty and most of the beauty of nuclear structure studies comes from the fact that the atomic nuclei consists of too many nucleons, and we can not solve exactly the quantum mechanical many-body problem. Therefore, models play the key role. But the fundamental structure models are based on different physical pictures.

The story is similar to the that of the ancient Indian parable, in which the blind men discover the elephant by touching it. Depending on which part their hand landed, they describe it as a thick snake, like a kind of fan, or pillar, like a tree trunk, etc.

Realising the magic proton and neutron numbers we think that the nucleus is like a small atom with shell structure. Looking at the rotational bands we get the impression that it is a microscopic liquid drop, which can rotate and vibrate. Studying its decay properties one says that it consists of smaller nuclei, like e.g alpha particles, so it has a cluster or molecular structure. In light of this variety it is an important question, how the fundamental models are connected to each other. What is their common features? How the elephant looks like, as a whole?

The basic discoveries along this line were made in 1958. Elliott showed that the quadrupole deformation and the collective rotation can be described within a spherical shell model, by the SU(3) symmetry of the valance nucleons [1]. Wildermuth and Kanellopoulos rewrote the shell model Hamiltonian into a cluster model one [2]. Bayman and Bohr could reinterprete this shell-cluster connection in terms of the U(3) symmetry [3]. Thus the common intersection of the three fundamental structure model was found in terms of the

$$\mathrm{U(3)} \supset \mathrm{SU(3)} \supset \mathrm{SO(3)} \qquad (1)$$

dynamical symmetry. (Though it was not expressed in this present-day language.) Anyway: the quadrupole collective bands and the cluster bands could be selected from the sea of the shell model basis according to their U(3) symmetries. It was a very important result, with far-reaching consequences.

The connection, however, was established only for a single shell problem; the Elliott model in its original form is a model of a single valence shell. But many nuclear phenomena go beyond a single shell framework, therefore, we need the relation for many major shells.

For the general case of arbitrary number of excitations the common intersection of the shell, collective and cluster models is again a U(3) dynamical symmetry, called Multiconfigurational Dynamical Symmetry (MUSY), which is composed of two U(3) subsymmetries:

$$\mathrm{U_s(3)} \otimes \mathrm{U_e(3)} \supset \mathrm{U(3)} \supset \mathrm{SU(3)} \supset \mathrm{SO(3)}, \qquad (2)$$

where $\mathrm{U_s(3)}$ refers to the space symmetry of the nucleons in a shell, when using the shell picture, or to the symmetry of the internal structure of the clusters in the cluster picture, while $\mathrm{U_e(3)}$ corresponds to shell excitations in both approaches [4].

It took a long time until the multi-shell generalization of the connection was found. Partly due to the fact that symmetry-governed multi-shell models had to be developed for the shell, collective and cluster picture.

In what follows first we describe the basic features of MUSY in Section 2, including a short review of the algebraic models that contribute to this composite symmetry. Then some applications are mentioned in Section 3, finally a brief summary is given and conclusions are drawn in Section 4.

2 Multiconfigurational dynamical symmetry

It is evident from the historical connection of 1958 that we need symmetry-based models for the multi-shell problem, which have to incorporate the antisymmetrization, in order to find the interrelation of the fundamental structure models. The antisymmetrization produce a finite (or full) overlap between the shell and (different) cluster configurations.

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In phenomenological models the wave functions of different configurations are orthogonal to each other. Therefore, we seek for models with microscopic model spaces and fully algebraic description, i.e. in which not only the basis states, but also the operators carry group symmetries.

2.1 Symmetry-adapted models

Shell picture The Symmetry Adapted No Core Shell Model (SA-NCSM) [5] provides us with a shell scheme of any number of excitation. In one of its versions the model space is a direct extension of the Elliott model. Both model and realistic interactions are used. With the latter ones the approach is an ab initio method, taking into account each nucleon, and performing first principle calculations.

The Semimicroscopic Algebraic Quartet Model (SAQM) [6] is practically a truncation of the SA-NCSM. It deals with quartet (N=Z=even) nuclei, and usually applies simple symmetric model Hamiltonians. The spin-isospin degrees of freedom are treated in terms of Wigner's $U^{ST}(4)$ formalism [7]. The quartet truncation specifies the Wigner-scalar (S=T=0) subspace of the general case.

The basis states of these models are given by the quantum numbers of (2). I.e the algebraic structure of the models is $U_s(3) \otimes U_e(3)$. This is a symmetry algebra, just like U(3) in the Elliott model.

A symmetry algebra can be embedded into a larger algebra, called dynamical algebra, which contains also a spectrum-generating part. For the shell model the dynamical algebra is Sp(3,R) [8]. An irreducible representation (irrep) of Sp(3,R) provides us with an infinite spectrum of a given parity. The symplectic shell model is a many major shell extension of the Elliott model, and the microscopic version of the collective model, too.

Collective picture The symplectic shell model has a simplified version, called contracted symplectic model [9, 10]. From the mathematical viewpoint the simplification means a simpler algebraic structure of $U_s(3) \otimes U_e(6)$, which is a compact algebra as opposed to the non-compact Sp(3,R). As a consequence its irreps have finite dimensions and orthonormal basis (unlike those of Sp(3,R)).

Another approach of the collective picture, worth mentioning here, concerns the search for stable deformation of nuclei. The usual way to do it is applying a structure model, and calculating the energy surface, where the local minima define the stable shapes. Based on the MUSY an alternative method can be followed. One can investigate the stability and the selfconsistency of the (united) U(3) of (2) [11]. The U(3) symmetry uniquely determines the quadrupole doformation thus one actually studies the stability and selfconsistency of deformation. One can perform Nilsson calculations, by changing systematically the deformation parameters, and the result is a stair-like function, showing stability regions, where also the selfconsistency requirement is fulfilled. Fig.1 shows the result for the ^{16}O nucleus.

This new method, called Stability and self-Consisteny of SU(3) Symmetry (SCS), gives usually good agreement

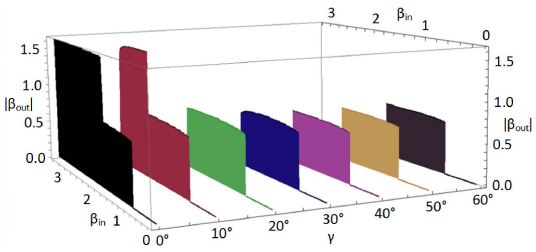

Figure 1. Shape isomers of the ^{16}O nucleus from the investigation of stability and selfconsistency of the U(3) symmetry (quadrupole deformation). In addition to the spherical ground state [4,4,4] a triaxial superdeformed state [12,4,0] and the linear alpha-chain [24,0,0] turnes out to be stable.

with the results of the traditional energy-minimum calculations. Its advantage is that having the U(3) symmetry of the shape isomers one can apply a selection rule for finding the allowed cluster configurations, i.e. reaction channels, which can produce the state in question. In these considerations not the simple U(3) symmetry plays the important role, rather its generalized version, called quasi-dynamical U(3) [12, 13], which remains valid even for many symmetry-breaking interactions.

Cluster picture In cluster models the basis with U(3) space (and $U^{ST}(4)$ spin-isospin) symmetry has been applied since 1958. Fully algebraic models, however, in which not only the states, but also the operators carry group symmetries were not available until Iachello invented the Vibron model [14]. That is a symmetry-based model of the dipole collectivity, which is relevant for the relative motion of the clusters. It has a U(4) dynamical group. It is extensively applied in molecular physics as well as in hadron spectroscopy. In nuclear physics it can describe a binary cluster system, if the clusters have no internal structure. Therefore, it is the simplest version of the Algebraic Cluster Model (ACM). When there are more clusters and/or internal degrees of freedom then the model has a larger group as dynamical group. ACM is a phenomenological approach in the sense that it does not contain the Pauli principle.

A Semimicroscopic Algebraic Cluster Model (SACM) was invented later on [15, 16], which combines the fully algebraic method of ACM with a microscopic model space, i.e. it includes the Pauli principle. In this approach the internal structure of the clusters is described by the Elliott model, while the relative motion is accounted for by the (modified) Vibron model. Therefore, its group structure for a binary configuration is $U^{ST}_{C1}(4) \otimes U_{C1}(3) \otimes U^{ST}_{C2}(4) \otimes U_{C2}(3) \otimes U_R(4)$. Please note, that the internal structures are characterized by symmetry algebras, while the relative motion is described by a dynamical one, incorporating a large scale of the relative motion (in line with the basic philosophy of the cluster picture). The SACM model space is identical with those of the fully microscopic cluster models (when they apply a $U^{ST}(4) \otimes U(3)$ basis).

In order to compare the performance of the ACM and SACM we consider here one of the simplest and best studied cluster system, i.e. ^{16}O+^{4}He. It has two closed-shell

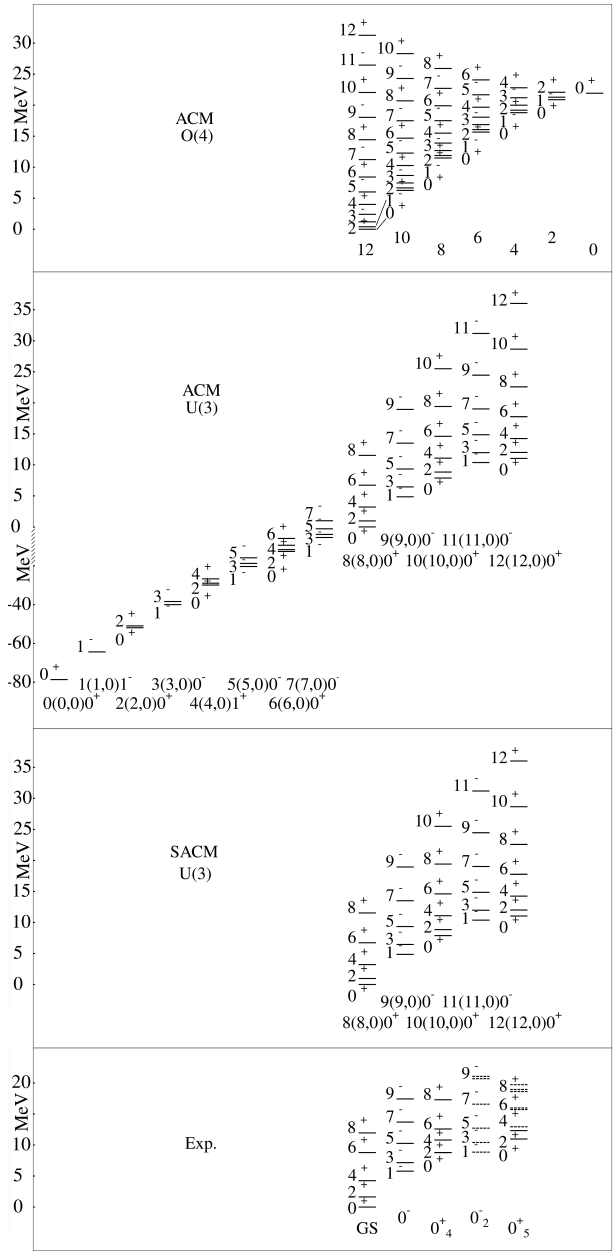

Figure 2. Spectra of dynamically symmetric Hamiltonians of the ^{16}O+^{4}He system (for N=12 bosons) in comparison with those bands of the of ^{20}Ne, which are considered to have core-plus-alpha cluster structure.

clusters, and several bands of ^{20}Ne are known to have this kind of structure. In Fig. 2. we show the experimental spectrum together with those of the dynamically symmetric Hamiltonians of the algebraic approaches. The dashed lines in the Exp. part indicate some uncertainty in the band-assignment. We also included the second negative parity band, though it is usually considered as a 1^- one, but it contains only states with natural spin-parity.

The dynamical symmetries of the Vibron model (ACM) are characterized by the group-chains: (i) U(4) ⊃ U(3) ⊃ O(3) and (ii) U(4) ⊃ O(4) ⊃ O(3), corresponding to the soft vibrator and rigid rotor limits (in the collective language) [14]. The Hamiltonians (up to two-body terms) for the dynamical symmetries of ACM are as follows: (i) $H = aC^{(1)}_{U3} + bC^{(2)}_{SU3} + dC^{(2)}_{SO3}$, and (ii) $H = cC^{(2)}_{SO4} + dC^{(2)}_{SO3}$.

From the comparison of the band structure of the dynamically symmetric Hamiltonians with that of the experimental observation, it is seen that (i) The experimental bands are much closer to the vibrational U(3) dynamical symmetry, than to the rigid rotor O(4) structure (of alternating parities). (ii) The microscopic model space of the SACM, which excludes the $n_R \leq 7$ Pauli forbidden states is realistic, while the small energy part of the phenomenological model is not.

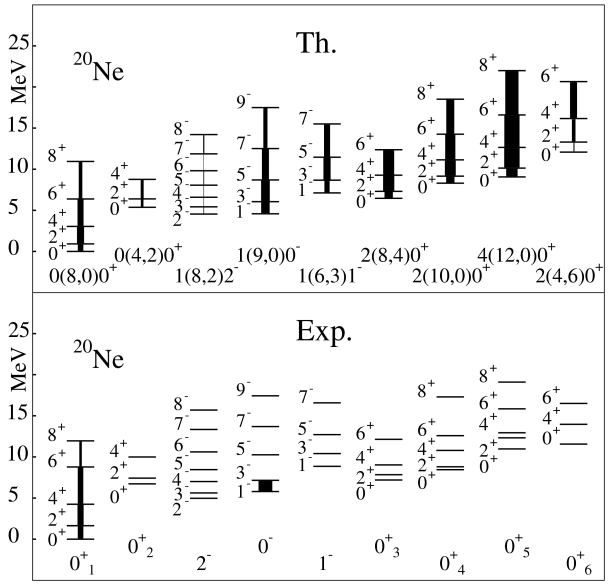

Figure 3. The MUSY spectrum in comparison with the experimental one of ^{20}Ne. All the well-established experimental bands of [17] are included containing both the core-plus-alpha structure of Fig. 1, and others.

2.2 Symmetry connections

As shown in the previous subsection, the multiplet structure of the shell, collective and cluster models are given by the group chain (2). This is the common (symmetry) section of these fundamental structure models for the multi major shell excitations. When applying algebraic spectrum generation, then (larger) dynamical groups appear as well: Sp(3,R) for shell, $U_s(3) \otimes U_e(6)$ for the contacted symplectic (collective) and $U_R(4)$ for the cluster picture. The (2) dynamical symmetry, being present in each configuration, is the first pillar of the MUSY.

MUSY is a composite symmetry in the sense that in addition to the common dynamical symmetry of the configurations, it has a further set of transformations, which take one configuration to the other. They act in the pseudo space of the particle numbers. The space part of them for N nucleons is described by the U(3N) ⊃ $U(3) \otimes U_p(N)$ group [4, 18], where U(3) is the space symmetry, as before, while $U_p(N)$ is the symmetry in the pseudo space of particle numbers.

2.3 Features of MUSY

The logical structure of MUSY, i.e. having a dynamical symmetry in each configuration, and transformations to connect the configurations resembles to that of the nuclear supersymmetry (SUSY). In SUSY both the bosonic sector (excitation quanta of collective motion) and the fermionic one (nucleons) have dynamical symmetries, and the supertransformations connect them. Therefore, interesting comparisons can be made between the two composite symmetries. E.g. one can study the role of interactions, which are invariant, and non-invariant with respect to the connecting transformations. Some studies of this kind have been carried out, and revealed remarkable features [19–21]. E.g. MUSY is usually applied with invariant Hamiltonians, like the SUSY of particle physics (contrary to the usual SUSY of nuclear spectroscopy). Nevertheless MUSY can be based also on non-invariant Hamiltonians, in which case it shows some logical similarity to that of the local gauge invariance.

The transformations in the pseudo space of particle indices is responsible for the overlap of the wave functions of different configurations. An especially interesting case is that of the total (100 %) overlap, which makes seemingly very different cluster and shell configurations identical with each other. They can be selected out easily, by making an expansion in terms of shell basis. Since both the eigenstates of different energy and of different U(3) symmetries are orthogonal to each other, when we have a U(3) basis with a single multiplicity in the shell space, then the corresponding configurations have identical wave functions (having only a single term in the expansion).

Another interesting aspect of MUSY is related to symmetry breaking. As mentioned above, the algebras of chain (2) are symmetry algebras. Nevertheless, different members of the chain play somewhat different roles. Only the SO(3) is an exact symmetry of the Hamiltonian, the larger algebras are dynamically broken by non-symmetric interactions. Just like the U(3) and SU(3) in the Elliott model.

On the other hand SO(3), the exact symmetry of the Hamiltonian is spontaneously broken [22]. It is an interesting historical fact that the spontaneous breaking of the rotational symmetry in the Elliott model has not been discussed until recently, in spite of the fact, that (i) it is generally accepted that the nuclear deformation is a result of the spontaneous breaking of rotational symmetry, and (ii) the Elliott model was the first to describe nuclear deformation and rotation in the spherical shell model.

It turns out that the scenario of the spontaneous breaking is very transparent in this model (and probably simpler than in others). It reads like that. (i) The separation of the fast (intrinsic) and slow (collective) degrees of freedom can be done exactly. (ii) The rotational symmetry spontaneously breaks in the eigenvalue equation of the intrinsic Hamiltonian, when there is degeneracy. (iii) Deformed shape and Goldstone bosons with zero mass show up. (iv) The symmetry restores when the complete Hamiltonian is considered.

Since the dynamical symmetry (2) of MUSY is a simple extension of that of the Elliott model, exactly the same scenario of the spontaneous breaking is valid for MUSY as well. Therefore, not only the quadrupole deformation, but also more complex shapes, like e.g. octupole or molecular configuration result from the spontaneous breaking of the rotational symmetry.

The dual breaking of symmetries of a group chain, as mentioned here for the Elliott model, i.e. dynamical breaking of the larger symmetries, and spontaneous breaking of the collective (rotational) symmetry is typical in algebraic models of nuclear structure [23].

3 Applications

Overview Several nuclei have been considered in order to determine their shape isomers with the SCS method: ^{28}Si in [24], ^{36}Ar in [25], ^{56}Ni in [26], alpha-like nuclei in between ^{12}C and ^{44}Ti in [27]. The results show good agreement with those of the energy-minimum calculations, based on Nilsson model [28], and alpha-cluster model [29–31] (when they are available).

The allowed binary cluster configurations have also been determined from the U(3) and UST(4) selection rules, therefore, we have a direct connection to the reaction channels in which they can be populated or they can decay.

In calculating the spectra we usually apply a simple dynamically symmetric Hamiltonian, which is invariant with respect to the transformations from one configuration to the other:

$$H = n\hbar\omega + aC^{(2)}_{SU3} + bC^{(3)}_{SU3} + d\frac{1}{2\theta}C^{(2)}_{SO3}. \qquad (3)$$

The eigenvalues of the Casimir operators: $\langle C^{(2)}_{SU3}\rangle = \lambda^2 + \mu^2 + \lambda\mu + 3(\lambda+\mu)$, $\langle C^{(3)}_{SU3}\rangle = (\lambda-\mu)(\lambda+2\mu+3)(2\lambda+\mu+3)$, $\langle C^{(2)}_{SO3}\rangle = L(L+1)$. The θ moment of inertia can be calculated either classically for a rigid ellipsoid, defined by the U(3) symmetry (in cylindrical approximation), or algebraically in terms of the invariant operators [32]. An illustrative example is given in Fig.3. The parameters are: $\hbar\omega = 6.8248$ MeV, $a = -0.1195$ MeV, $b = -0.00025$ MeV, $d = 0.7905$. and the moments of inertia are those of the ellipsoids.

When calculating the electric quadrupole transitions, we usually apply again an operator, which is invariant under the transformations between the configurations, and has a single parameter (effective charge), which can be fitted to the best-known low-energy experimental value.

Case studies Here we mention some examples in more details in order to illustrate the interaction between the model calculation and the experimental observation.

The hyperdeformed band of ^{36}Ar was predicted first from an alpha-cluster calculation [33], and its possible binary clusterizations (reaction channels) were determined in [34]. Its existence and moment of inertia was also predicted from the SCS calculation in [25], and a candidate band was observed experimentally in [35], in good agreement with the theoretical predictions. The energy spectrum of the ground, superdeformed and hyperdeformed valleys are calculated in a unified way in [36].

In searching for the superdeformed (SD) state of the ^{28}Si nucleus an SCS calculation predicted its deformation, moment of inertia and preferred reaction channels [24]. A simultaneous experimental study suggested a candidate state with good agreement with the prediction [37]. The spectrum of the well-known bands was described by a MUSY Hamiltonian (in terms of the quartet model), and this calculation gave a parameter-free prediction without ambiguity for the detailed high-lying ^{16}O+^{12}C cluster spectrum [38]. It shows a close similarity to the fine-resolution spectra of resonances from the ^{16}O+^{12}C reactions. It is especially remarkable, because not only that it was described together with the low-energy (shell) spectrum, but it was obtained as an extrapolation from it. This calculation answered the long-standing question: the resonances of the ^{16}O+^{12}C reaction populate the spectrum of the second minimum, i.e. superdeformed valley.

The band-head of the SD candidate band was missing from the experimental side. Therefore, inelastic alpha-scattering was studied in order to populate the high-lying 0^+ states in ^{28}Si. The observed distribution of these highly excited states turned out to be in good agreement with the 0^+ states of the ^{24}Mg+^{4}He cluster configuration [39]. This calculation applied again the MUSY Hamiltonian fitted to the low-lying shell states, i.e. the distribution of the high-lying cluster states was obtained as a parameter-free prediction.

Then a measurement of the $4^+ \rightarrow 2^+$ B(E2) value [40] turned out to be much smaller than the prediction of the Antisymmetrized Molecular Dynamics ($\leq$ 43 W.u. vs. 188 W.u.) [41], thus the experimentalist rejected the candidate SD band. The U(3) prediction, however, is considerable smaller (56.8 W.u.) [42]. Therefore, the question is still open, and the band can be a candidate for superdeformation. (The same kind of prediction in the ^{36}Ar case turned out to be in a good agreement with the observation.)

An investigation of the ^{24}Mg nucleus was inspired by the discovery of some new 0^+ states around the ^{12}C+^{12}C threshold [43]. It was suggested that these resonances have ^{12}C+^{12}C cluster structure, and play a similar role in the carbon burning in the stars, like the Hoyle state in the helium burning. The ^{20}Ne+^{4}He decay was considered as a further indication for this interpretation. In [44], this hypothesis was checked quantitatively from the theoretical side. In particular, the distribution of the 0^+ states was studied in a very wide energy range, both in the quartet model, and in the relevant cluster configurations. The reliability of the MUSY calculation is shown by the fact, that the energy spectrum from the ground state up to the (theoretically predicted) very largely deformed states was in good agreement with the observation, just like the predicted ^{12}C+^{12}C cluster spectrum with the experimentally found resonances. The conclusion is that the density of the simple ^{12}C+^{12}C cluster states (with intrinsic ground state clusters) with 0^+ spin-parity is much smaller, than the experimental observation. Therefore, the detected states more probably correspond to fragmented cluster states (coupled to background states, e.g. quartet excitations of ^{24}Mg).

As for the alpha-like nuclei (in between ^{12}C and ^{44}Ti) are concerned, not only their shape isomers and possible clusterizations were determined in [27], but also their energy spectra. It is remarkable that a simple dynamically symmetric Hamiltonian could reproduce their complete spectra obtained from Bloch-Brink model calculation with a very different force.

4 Summary and conclusions

The multiconfigurational dynamical symmetry is the common intersection of the shell, collective and cluster models for the many major shell problem. Therefore, it can give a unified description of phenomena and spectra scattered along the axes of the excitation energy, and deformation, and coming from different reactions. Due to the same reason it has a considerable predictive power.

MUSY is a composite symmetry in the sense that (i) each configuration has the same U(3) $\otimes$ U(3) $\supset$ U(3) dynamical symmetry, i.e. they have a common multiplet structure, and furthermore (ii) there is a set of transformations which connect the configurations to each other.

The characteristic features of the application of MUSY are as follows. (i) Usually it involves huge model space and simple symmetric operators. (ii) A strong emphases is put on the interrelation of different models, from the viewpoint of the investigated phenomenon. This is somewhat different from the traditional approach of describing the data within a single model, without paying much attention to other aspects.

Though it is a relatively young symmetry, some of its predictions seem to be verified already from the experimental side.

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