EPJ Web of Conferences **314**, 00025 (2024) https://doi.org/10.1051/epjconf/202431400025
QCD@Work 2024

Topology of the large-N expansion in SU(N) Yang-Mills theory and spin-statistics theorem

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Abstract. Recently, we computed the generating functional of Euclidean asymptotic correlators at short-distance of single-trace twist-2 operators in large-N SU(N) Yang-Mills (YM) theory to the leading-nonplanar order. Remarkably, it has the structure of the logarithm of a functional determinant, but with the sign opposite to the one arising from the spin-statistics theorem for the glueballs. To solve the sign puzzle, we reconsider the proof that in 't Hooft large-N expansion of YM theory the leading-nonplanar contribution to the generating functional consists of the sum over punctures of n-punctured tori. We discover that for twist-2 operators it contains – in addition to the n-punctured tori – the normalization of tori with $1 \leq p \leq n$ pinches and $n - p$ punctures. Once the existence of the new sector is taken into account, the violation of the spin-statistics theorem disappears. Besides, the new sector contributes trivially to the nonperturbative S matrix because – for example – the n-pinched torus represents nonperturbatively a loop of n glueball propagators with no external leg. This opens the way for an exact solution limited to the new sector that may be solvable thanks to the vanishing S matrix.

It has been known for more than forty years that U(N) Yang-Mills (YM) theory admits 't Hooft large-N topological expansion [1] for the n-point connected correlators of gauge-invariant single-trace operators. In the present paper [1] we work out the SU(N) case, almost literally summarizing [2]. In 't Hooft large-N expansion, the Feynman diagrams in double-line representation [1] contributing to the above correlators – after a suitable gluing of reversely oriented lines – are topologically classified [1, 3] by the sum on the genus g of n-punctured closed Riemann surfaces, each topology being weighted by N^χ, with $\chi = 2-2g-n$ the Euler characteristic of the Riemann surface. 't Hooft expansion extends to large-N QCD [1, 3] – and generally to large-N gauge theories – by eventually including punctured Riemann surfaces with boundaries. It exactly matches the topology and weights [1, 3] of a string theory [4], with closed-string coupling $g_s = \frac{1}{N}$. This matching has been advocated for the existence [1, 3, 5] of a nonperturbative string solution of large-N YM theory, QCD and, more generally, gauge theories, where – in the asymptotically free case – the dimensionful parameter of the string theory, i.e. the string tension, is identified [6] with the square of the renormalization-group (RG) invariant scale Λ_{RG} of the gauge theory. In the supposed string solution of large-N QCD glueballs are excitations of closed strings, mesons of open strings, and the above

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[1]M. Bochicchio invited talk at the conference QCD@work2024.

Riemann surfaces arise as their world-sheets – the new ingredient of the canonical string solution [6] being the existence of a conformal field theory living on the string world-sheets that is employed to compute the S-matrix amplitudes [4] and possibly the correlators [7] of the gauge theory. Actually, the existence of the supposed canonical string solution of (YM theory) QCD is constrained [6] by the large-N nonperturbative (non-)renormalization properties [8] of $\Lambda_{(YM)QCD}$. By assuming 't Hooft topological expansion, the generating functional $\mathcal{W}^E[J_O] = \log \mathcal{Z}^E[J_O]$ of Euclidean connected correlators of single-trace operators O in SU(N) YM theory, with

$$\mathcal{Z}^E[J_O] = \frac{1}{\mathcal{Z}^E} \int \mathcal{D}A\, e^{-S_{YM}+\sum_s \int J_{O_s} O_s}\,, \tag{1}$$

reads $\mathcal{W}^E[J_O] = \mathcal{W}^E_{\text{sphere}}[J_O] + \mathcal{W}^E_{\text{torus}}[J_O] + \cdots$ (Fig. 1). Nonperturbatively, $\mathcal{W}^E_{\text{sphere}}[J_O]$,

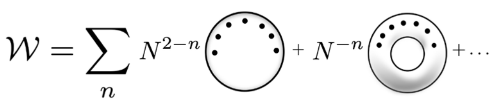

Figure 1. 't Hooft topological expansion of the generating functional that includes n-punctured spheres and tori.

which perturbatively is the ('t Hooft-)planar contribution [1], is a sum of tree diagrams involving glueball propagators and vertices, while $\mathcal{W}^E_{\text{torus}}[J_O]$, which perturbatively is the leading-non('t Hooft-)planar contribution, is a sum of glueball one-loop diagrams. Nonperturbatively, $\mathcal{W}^E_{\text{torus}}[J_O]$ should have the structure of the logarithm of a functional determinant [9]. Indeed, in the yet-to-come nonperturbative solution of large-N YM theory, the very same correlators should be computed by the correlators of a glueball field Φ with an infinite number of components, the corresponding generating functional being schematically [9]

$$\mathcal{Z}^E_{\text{glueball}}[J] = \frac{1}{\mathcal{Z}^E_{\text{glueball}}} \int \mathcal{D}\Phi\, e^{-S_{\text{glueball}}(\Phi)+\int \Phi *_1 J}\,, \tag{2}$$

with $S_{\text{glueball}}(\Phi) = \frac{1}{2}\int \Phi *_2 (-\Delta + M^2)\Phi + \frac{1}{N}\frac{1}{3}\Phi *_3 \Phi *_3 \Phi + \cdots$, where $*_2$ and $*_1$ are fixed below, the ellipses and $*_3$ respectively stand for n-glueball vertices with $n > 3$ and some presently unknown operation on the glueball fields. Hence, nonperturbatively the connected generating functional $\mathcal{W}^E_{\text{glueball}}[J] = \log \mathcal{Z}^E_{\text{glueball}}[J]$ reads to one loop of glueballs [9]

$$\mathcal{W}^E_{\text{glueball}}[J] = -S_{\text{glueball}}(\Phi_J) + \int \Phi_J *_1 J + \cdots - \frac{1}{2}\log\mathrm{Det}\left(*_2(-\Delta + M^2) + \frac{1}{N} *_3 \Phi_J *_3 + \cdots\right), \tag{3}$$

with Φ_J determined by $\left.\frac{\delta S_{\text{glueball}}}{\delta\Phi}\right|_{\Phi_J} = *_1 J$. The minus sign in front of $\log\mathrm{Det}$ in $\mathcal{W}^E_{\text{glueball}}[J]$ arises from the spin-statistics theorem [10], since all the gauge-invariant glueball interpolating fields have integer spin, and thus the glueballs should be bosons. The dictionary between $\mathcal{W}^E[J_O]$ and $\mathcal{W}^E_{\text{glueball}}[J]$ arises from matching the corresponding spectral representations – as a sum of free propagators with residues R_{sm} – for the 2-point correlators [11] of O_s at $N = \infty$ that, by fixing $*_2$ according to the canonical normalization of the glueball kinetic term, uniquely determines the coupling of J to the tower of glueball fields $\Phi *_1 J = \sum_{sm} \Phi_{sm} \sqrt{R_{sm}} J_s$. Until recently nothing has been quantitatively known on $\mathcal{W}^E[J_O]$ and $\mathcal{W}^E_{\text{glueball}}[J]$. Actually, since the early days of large-N QCD, it has been qualitatively known that asymptotic freedom (AF) applies to large-N correlators, and specifically to the nonperturbative 2-point correlators [12, 13]. Accordingly, the ultraviolet (UV), i.e. short-distance, constraints on the large-N spectral representation of 2-point correlators implied by AF were quantitatively investigated for multiplicatively renormalizable operators [14] and twist-2 operators that mix by renormalization [15]. More recently, we computed the UV asymptotics of the generating functional $\mathcal{W}^E[J_O]$ of Euclidean correlators of twist-2 operators in large-N YM theory [16] that follows from RG, Callan-Symanzik equation and AF. For simplicity, we report just a particular case of our calculation that involves balanced collinear twist-2 operators [16, 17] with even spin and maximal-spin component in the p_+ direction in

Minkowskian space-time. In the light-cone gauge $A_+ = 0$ they read [16, 17]

$$\mathbb{O}_s = \frac{1}{2N}\bar{A}^a(x)\mathcal{Y}^{\frac{5}{2}}_{s-2}(\overrightarrow{\partial}_+, \overleftarrow{\partial}_+)A^a(x), \qquad \mathcal{Y}^{\frac{5}{2}}_{s-2}(\overrightarrow{\partial}_+, \overleftarrow{\partial}_+) = \overleftarrow{\partial}_+(i\overrightarrow{\partial}_+ + i\overleftarrow{\partial}_+)^{s-2}C^{\frac{5}{2}}_{s-2}\left(\frac{\overrightarrow{\partial}_+ - \overleftarrow{\partial}_+}{\overrightarrow{\partial}_+ + \overleftarrow{\partial}_+}\right)\overrightarrow{\partial}_+\,, \tag{4}$$

where $s = 2, 4, 6, \ldots$, the sum over repeated color indices, $a = 1, 2, \ldots N^2 - 1$, is understood, and $C^{\frac{5}{2}}_{s-2}$ are Gegenbauer polynomials. Our calculation was performed in three steps that we briefly recall. First, we computed to the lowest perturbative order – directly from its functional-integral definition as a Gaussian integral

$$\mathcal{Z}_{\mathrm{conf}}[J_{\mathbb{O}}] = \frac{1}{Z}\int \mathcal{D}A\mathcal{D}\bar{A}\, e^{\int -i\bar{A}^a\Box A^a + \sum_s J_{\mathbb{O}_s}\mathbb{O}_s\, d^4x} \tag{5}$$

the generating functional of the connected conformal correlators in Minkowskian space-time and – by analytical continuation – the corresponding object in Euclidean space-time [16], respectively

$$\mathcal{W}_{\mathrm{conf}}[J_{\mathbb{O}}] = -(N^2-1)\log\mathrm{Det}\left(\mathbb{I} + \tfrac{1}{2}i\Box^{-1}\tfrac{J_{\mathbb{O}_s}}{N}\otimes\mathcal{Y}^{\frac{5}{2}}_{s-2}\right), \qquad \mathcal{W}^E_{\mathrm{conf}}[J_{\mathbb{O}^E}] = -(N^2-1)\log\mathrm{Det}\left(\mathbb{I} + \tfrac{1}{2}\Delta^{-1}\tfrac{J_{\mathbb{O}^E_s}}{N}\otimes\mathcal{Y}^{E\frac{5}{2}}_{s-2}\right), \tag{6}$$

where $\mathbb{I}$ is the identity in space-time, the sum over repeated s indices is understood and $\mathcal{Y}^{E\frac{5}{2}}_{s-2}$ is obtained from $\mathcal{Y}^{\frac{5}{2}}_{s-2}$ by the substitution $\partial_+ \to i\partial_z$ [16, 17]. Second, we worked out the UV asymptotics, as all the coordinates are uniformly rescaled by a factor $\lambda \to 0$, of renormalized correlators of collinear twist-2 operators that mix by renormalization with derivatives along the maximal-spin direction of collinear twist-2 operators of lower spin and same canonical dimensions [16]

$$G^{(n)}_{k_1\ldots k_n}(\lambda x_1, \ldots, \lambda x_n; \mu, g(\mu)) = \sum_{j_1\cdots j_n} Z_{k_1j_1}(\lambda)\ldots Z_{k_nj_n}(\lambda)\,\lambda^{-\sum_{i=1}^n(s_{j_i}+2)}G^{(n)}_{j_1\ldots j_n}(x_1, \ldots, x_n; \mu, g(\frac{\mu}{\lambda}))\,, \tag{7}$$

reducing the above mixing to the multiplicatively renormalizable case [16]

$$G^{(n)}_{j_1\cdots j_n}(\lambda x_1, \ldots, \lambda x_n; \mu, g(\mu)) = Z_{\mathbb{O}_{j_1}}(\lambda)\ldots Z_{\mathbb{O}_{j_n}}(\lambda)\,\lambda^{-\sum_{i=1}^n(s_{j_i}+2)}G^{(n)}_{j_1\ldots j_n}(x_1, \ldots, x_n; \mu, g(\frac{\mu}{\lambda}))\,, \tag{8}$$

thanks to the inductive construction to all orders of perturbation theory [16, 18] of a finite change of renormalization scheme for the renormalized operators, where $Z_{sj}(\lambda) = \delta_{sj}Z_{\mathbb{O}_s}(\lambda) = \delta_{sj}(\frac{g(\mu)}{g(\frac{\mu}{\lambda})})^{\frac{\gamma_{0\mathbb{O}_s}}{\beta_0}}$ is diagonal and one-loop exact [18], and independent of N [16], while in any scheme

$$G^{(n)}_{j_1\ldots j_n}(x_1, \ldots, x_n; \mu, g(\frac{\mu}{\lambda})) \sim G^{(n)}_{\mathrm{conf}\,j_1\ldots j_n}(x_1, \ldots, x_n) + g^2(\frac{\mu}{\lambda})\,\mathcal{G}^{(n,2)}_{j_1\ldots j_n}(x_1, \ldots, x_n; \mu) + \cdots\,. \tag{9}$$

Consequently [16], since $g(\frac{\mu}{\lambda}) \to 0$ as $\lambda \to 0$ by AF,

$$G^{(n)}_{j_1\ldots j_n}(\lambda x_1, \ldots, \lambda x_n; \mu, g(\mu)) \sim \frac{Z_{\mathbb{O}_{j_1}}(\lambda)\ldots Z_{\mathbb{O}_{j_n}}(\lambda)}{\lambda^{\sum_{i=1}^n(s_{j_i}+2)}}G^{(n)}_{\mathrm{conf}\,j_1\ldots j_n}(x_1, \ldots, x_n)\,. \tag{10}$$

Third, by Eq. (10) we lifted $\mathcal{W}^E_{\mathrm{conf}}[J_{\mathbb{O}^E}]$ to the generating functional of RG-improved UV asymptotic correlators $\mathcal{W}^E_{\mathrm{asym}}[J_{\mathbb{O}^E}, \lambda]$ that inherits the very same structure of the logarithm of a functional determinant

$$\mathcal{W}^E_{\mathrm{asym}}[J_{\mathbb{O}^E}, \lambda] = -(N^2-1)\log\mathrm{Det}\left(\mathbb{I} + \frac{1}{2}\frac{Z_{\mathbb{O}_s}(\lambda)}{\lambda^{s+2}}\Delta^{-1}\frac{J_{\mathbb{O}^E_s}}{N}\otimes\mathcal{Y}^{E\frac{5}{2}}_{s-2}\right). \tag{11}$$

The conformal generating functional admits the large-N expansion that trivially follows from Eq. (6), $\mathcal{W}_{\mathrm{conf}}[J_{\mathbb{O}^E}] = \mathcal{W}_{\mathrm{conf\,sphere}}[J_{\mathbb{O}^E}] + \mathcal{W}_{\mathrm{conf\,torus}}[J_{\mathbb{O}^E}]$, and the asymptotic generating functional as well $\mathcal{W}^E_{\mathrm{asym}}[J_{\mathbb{O}^E}, \lambda] = \mathcal{W}^E_{\mathrm{asym\,sphere}}[J_{\mathbb{O}^E}, \lambda] + \mathcal{W}^E_{\mathrm{asym\,torus}}[J_{\mathbb{O}^E}, \lambda]$, with

$$\mathcal{W}^E_{\mathrm{asym\,torus}}[J_{\mathbb{O}^E}, \lambda] = +\log\mathrm{Det}\left(\mathbb{I} + \frac{1}{2}\frac{Z_{\mathbb{O}_s}(\lambda)}{\lambda^{s+2}}\Delta^{-1}\frac{J_{\mathbb{O}^E_s}}{N}\otimes\mathcal{Y}^{E\frac{5}{2}}_{s-2}\right). \tag{12}$$

Remarkably, the above equation reproduces the log Det structure of the glueball one-loop generating functional in Eq. (3), which it should be UV asymptotic to thanks to the AF, but with the sign opposite to the one arising from the spin-statistics theorem for the glueballs. Yet, we verify the correctness of the above sign, which is inherited from Eqs. (6) and (11), according to the sign arising from the spin-statistics theorem for gluons in one-loop perturbation theory, and the fact that in the SU(N) theory there are exactly $N^2 - 1$ gluons. The aim of the present paper is to solve the sign puzzle and discuss the implications of its solution. Essentially, there are two possible way-outs: Either the spin-statistics theorem is violated or 't Hooft topological expansion (Fig. 1) and the corresponding effective action in Eq. (3) need considerable refinements for the correlators of twist-2 operators. As we shall show momentarily, only the second alternative applies in large-N YM theory.

In fact, the spin-statistics theorem applies in full generality only to theories with a finite number of local fields [19–22]. If the number of fields is infinite, there exist rigorous counterexamples [19–21] to the spin-statistics theorem. They involve massive infinite-dimensional representations of the Lorentz group that are the relevant ones in YM theory because of its mass gap. Yet, they have infinite mass degeneracy, since they decompose [20, 21], according to Wigner theorem [23], into the sum of irreducible representations of the Poincaré group corresponding to an infinite number of particles of any spin, all having the same mass. In large-N YM theory this would imply the existence of vertical Regge trajectories that is hardly acceptable both theoretically and numerically because of contrary evidence from analysis [24] of lattice calculations.

Hence, it remains the second alternative. In order to understand why new topologies arise in the generating functional of correlators of twist-2 operators, we have reconsidered the proof of 't Hooft topological expansion in the U(N) versus SU(N) YM theory. The delicate point of the proof in the SU(N) theory is that – contrary to the original U(N) case [1] – the gluon propagator has a leading and subleading $\frac{1}{N}$ contribution in its double-line representation [25, 26]: $\langle A^i_{\,j} A^k_{\,l}\rangle \propto I^{ik}_{jl} - P^{ik}_{jl}$, with $A^i_{\,j} = A^a(\lambda^a)^i_{\,j}$, λ^a in the fundamental representation, $\mathrm{Tr}(\lambda^a\lambda^b) = \frac{1}{2}\delta^{ab}$, $[\lambda^a, \lambda^b] = if^{abc}\lambda^c$, $I^{ik}_{jl} = \delta^i_{\,l}\delta^k_{\,j}$ the components of the identity I in the u(N) Lie algebra, where the product of two matrices, B and C, is defined by $(BC)^{im}_{jn} = B^{ik}_{jl}C^{lm}_{kn}$, with the color trace $\mathrm{Tr}\,B = B^{ij}_{ji}$, $P^{ik}_{jl} = \frac{1}{N}\delta^i_{\,j}\delta^k_{\,l}$ the components of the u(1) projector P, and $i, l, j, k = 1, \ldots, N$. Yet, the subleading propagator, if it is attached to the vertices of the action, $V_3 \propto \frac{g}{\sqrt{N}} f^{abc}$ and $V_4 \propto \frac{g^2}{N} f^{abe} f^{ecd}$, does not contribute [25, 26]. This is rather obvious [26], since the subleading propagator is a u(1) contribution, while the action vertices are purely nonabelian. Therefore, in the SU(N) theory the topology of connected vacuum diagrams containing vertex insertions matches 't Hooft topological expansion with weights N^χ, where the gluon propagator is just the leading one [25, 26]. At this point, generalizing the structure of V_3 and V_4, we extend the above argument to connected correlators of single-trace operators provided that the corresponding local vertices are proportional to a matrix product [26] of $(T^a)^b_c = -if^{abc}$, $(T^{a_2} \ldots T^{a_{n-1}})^{a_1}_{a_n} = 2\,\mathrm{Tr}(\lambda^{a_1}[\lambda^{a_2}, \ldots [\lambda^{a_{n-1}}, \lambda^{a_n}] \ldots])$ that requires at least 3-gluon operators in perturbation theory. Indeed, the above vertices in double line have no $\frac{1}{N}$ correction [26] and their contraction with the subleading propagator vanishes because they involve nested commutators. Therefore, there is a vast – but quite specific – class of single-trace operators which the 't Hooft topological expansion applies to in SU(N) YM theory. Yet, we point out that the above proof does not apply to 2-gluon operators, and specifically to twist-2 operators. Indeed, in this case – even in the adjoint representation – their local vertex involves δ^{ab}, as opposed to f^{abc}, and contains a $\frac{1}{N}$ correction (Fig. 2a), exactly as the gluon propagator does. In fact, it has been suggested to consider the U(N) [1, 5, 25] – rather than the SU(N) – theory, where perturbatively no subleading correction both to the gluon propagator and (bare) 2-gluon vertices occur – though gauge-invariant single-trace twist-2 operators have not been specifically discussed. Nevertheless, this seemingly easy way-out is pointless for asymptotically free theories that allow us to resum the (renormalized) RG-improved generating functional into the logarithm of a functional determinant

$$\begin{aligned}\mathcal{W}^E_{U(N)\mathrm{asym}}[J_{\mathbb{O}}, \lambda] = &- (N^2 - 1) \log \mathrm{Det}\left(\mathbb{I} + \frac{1}{2}\frac{Z_{\mathbb{O}_s}(\lambda)}{\lambda^{s+2}}\Delta^{-1}\frac{J_{\mathbb{O}^E_s}}{N} \otimes \mathcal{Y}^{E\frac{5}{2}}_{s-2}\right)\\ &- \log \mathrm{Det}\left(\mathbb{I} + \frac{1}{2}\frac{1}{\lambda^{s+2}}\Delta^{-1}\frac{J_{\mathbb{O}^E_s}}{N} \otimes \mathcal{Y}^{E\frac{5}{2}}_{s-2}\right), \end{aligned} \tag{13}$$

while the U(1) contribution in the second line decouples and is exactly conformal and not only asymptotic because the U(1) theory is free. Indeed, asymptotically in the UV the above result is dominated by only the SU(N) contribution for the correlators of all the twist-2 operators but the stress-energy tensor, since for them $\gamma_{0\mathbb{O}_s}$ is positive [16, 27]. Hence, the sign problem for the U(N) theory is exactly the same as for the SU(N) theory. To solve the sign problem in the SU(N) theory, we rewrite Eq. (6) identically as

$$\begin{aligned}\mathcal{W}^E_{\text{conf}}[J_{\mathbb{O}^E}] &= -\log \mathrm{Det}\left(\mathcal{I} + \frac{1}{2}(I-P)\Delta^{-1}\frac{J_{\mathbb{O}^E_s}}{N}\otimes \mathcal{Y}^{E\frac{5}{2}}_{s-2}\right)\\ &= -\log \mathrm{Det}\left[\left(\mathcal{I} + \frac{1}{2}\Delta^{-1}I\frac{J_{\mathbb{O}^E_s}}{N}\otimes \mathcal{Y}^{E\frac{5}{2}}_{s-2}\right)\left(\mathcal{I} - \frac{1}{2}\left(\mathcal{I} + \frac{1}{2}\Delta^{-1}I\frac{J_{\mathbb{O}^E_s}}{N}\otimes \mathcal{Y}^{E\frac{5}{2}}_{s-2}\right)^{-1}\Delta^{-1}P\frac{J_{\mathbb{O}^E_s}}{N}\otimes \mathcal{Y}^{E\frac{5}{2}}_{s-2}\right)\right], \end{aligned} \tag{14}$$

where the term containing $I-P$ involves the SU(N) propagator, $\mathcal{I}$ is the identity in both color and space-time and the first equality above follows by noticing that the color trace Tr in the loop expansion of Eq. (14) containing the insertion of n sources $J_{\mathbb{O}^E_s}$ produces the overall factor $\mathrm{Tr}(I-P)^n = \mathrm{Tr}(I-P) = N^2-1$ that occurs in Eq. (6). For keeping 't Hooft double-line representation, which only involves the leading propagator, also beyond the planar limit of the SU(N) theory, we transfer in Eq. (14) the $\frac{1}{N}$ dependence from the propagator to the local vertex for 2-gluon operators (Fig. 2a) so that

$$\begin{aligned}\mathcal{W}^E_{\text{conf}}[J_{\mathbb{O}^E}] = &-\log \mathrm{Det}\left(\mathcal{I} + \frac{1}{2}\Delta^{-1}I\frac{J_{\mathbb{O}^E_s}}{N}\otimes \mathcal{Y}^{E\frac{5}{2}}_{s-2}\right)\\ &-\log \mathrm{Det}\left(\mathcal{I} - \frac{1}{2}\left(\mathcal{I} + \frac{1}{2}\Delta^{-1}I\frac{J_{\mathbb{O}^E_s}}{N}\otimes \mathcal{Y}^{E\frac{5}{2}}_{s-2}\right)^{-1}\Delta^{-1}P\frac{J_{\mathbb{O}^E_s}}{N}\otimes \mathcal{Y}^{E\frac{5}{2}}_{s-2}\right). \end{aligned} \tag{15}$$

The first log Det above is the ('t Hooft-)planar contribution – involving the leading local vertex (Fig. 2a) – and the second log Det the leading-nonplanar one – involving at least one subleading local vertex (Fig. 2a) carrying the factor P. The latter gives rise to new topologies. For example, for 2-point correlators (Fig. 2b), by the standard 't Hooft gluing of reversely oriented lines, the first diagram in Fig. 2b – that

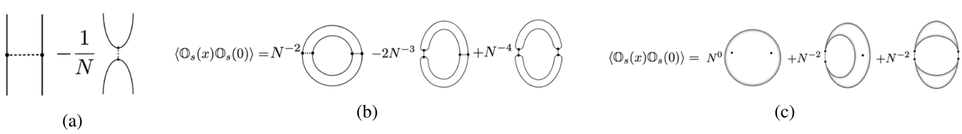

Figure 2. (a) Local vertex for 2-gluon operators in double-line. The dotted lines identify the punctures. (b) 2-point correlators of twist-2 operators to the leading perturbative order, with weights as in Fig. 3a. (c) Topology and weights of diagrams in Fig. 2b.

is the planar contribution – leads to a 2-punctured sphere and the remaining ones – by representing a 2-punctured disk as an infinite strip and gluing the opposite edges of the strip to get an infinite cylinder, i.e. a 2-punctured sphere – to possibly disconnected punctured spheres with at least two punctures pairwise identified (Fig. 2c). By generalizing the new diagrams in Fig. 2b, the leading-nonplanar contributions to the second log Det in Eq. (15) involve the new topological sector (Fig. 3a). After the

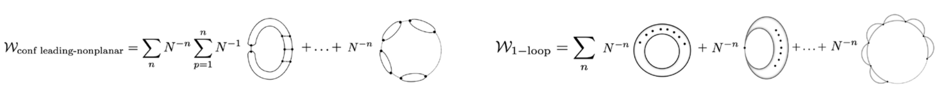

Figure 3. (a) The refined perturbative expansion of the conformal leading-nonplanar generating functional, with n pairs of punctures and p insertions of P, which give rise to the weights N^{-p} compensated for by the color traces. (b) After gluing reversely oriented lines, the new topologies arise from the normalization, obtained by cutting the pinches, of p-pinched tori with $n-p$ punctures.

RG improvement, the leading-nonplanar asymptotic generating functional reads

$$\begin{aligned}\mathcal{W}^E_{\text{asym nonplanar}}[J_{\mathbb{O}^E}, \lambda] &= -\log \mathrm{Det}\left(\mathcal{I} - \frac{1}{2}\left(\mathcal{I} + \frac{1}{2}\frac{Z_{\mathbb{O}_s}(\lambda)}{\lambda^{s+2}}\Delta^{-1}I\frac{J_{\mathbb{O}^E_s}}{N}\otimes \mathcal{Y}^{E\frac{5}{2}}_{s-2}\right)^{-1}\frac{Z_{\mathbb{O}_s}(\lambda)}{\lambda^{s+2}}\Delta^{-1}P\frac{J_{\mathbb{O}^E_s}}{N}\otimes \mathcal{Y}^{E\frac{5}{2}}_{s-2}\right)\\ &= +\log \mathrm{Det}\left(\mathbb{I} + \frac{1}{2}\frac{Z_{\mathbb{O}_s}(\lambda)}{\lambda^{s+2}}\Delta^{-1}\frac{J_{\mathbb{O}^E_s}}{N}\otimes \mathcal{Y}^{E\frac{5}{2}}_{s-2}\right). \end{aligned} \tag{16}$$

Remarkably, now the overall sign of the first log Det in Eq. (16) is consistent with the bosonic statistics for the glueballs, but at the price of introducing a refined topological expansion where, by generalizing the new topologies in Fig. 2c for each n, in addition to the n-punctured tori, punctured spheres that are the normalization [28, 29] of p-pinched and $n-p$-punctured tori (Fig. 3b) arise, with $1 \leq p \leq n$. Besides, the punctured smooth tori are suppressed in perturbation theory with respect to the pinched ones, since the smooth tori inevitably involve the action vertices that carry powers of 't Hooft gauge coupling. Yet, in the RG-improved generating functional the smooth tori are essential to provide the renormalization factors in Eq. (16) due to the anomalous dimensions also for the new topologies that do not couple perturbatively to the action vertices V_3 and V_4, but mix with the punctured smooth tori by perturbative renormalization, since they have the same weights.

We provide now a nonperturbative interpretation of our refined topological expansion for twist-2 operators – that, incidentally, nonperturbatively explains the above mixing – in terms of the effective theory of glueballs that is closely related – though not exactly coincident – to the dual-graph representation of Riemann surfaces [28, 29] recalled below. It has been known for more than forty years that, in the effective theory of glueballs, punctured spheres correspond to glueball tree graphs [11, 12]. Specifically, the sphere with two punctures corresponds to an infinite sum of glueball propagators [11], while the sphere with three punctures corresponds in Minkowskian space-time to vertices that involve sums of three or two glueball poles [12] (Fig. 4a). For simplicity, in the following we skip n-glueball vertices with $n > 3$ as in Eq. (3). Incidentally, we point out that the last graph in Fig. 4a contributes zero to the S matrix because of the missing glueball external leg. Analogously, the 2-punctured torus may contribute, in addition to glueball one-loop two-leg graphs, also graphs with only one or zero external glueball legs (Fig. 4b) and, similarly, our new topologies may contribute the graphs in Fig. 4c. Incidentally, the partial matching of the graphs in Fig. 4c with the ones in Fig. 4b nonperturbatively explains the aforementioned mixing by renormalization. Analogous statements hold for the graphs in Figs. 3b and 5. Both graphs in Fig. 4c contribute zero to the S matrix as well. More generally, all the

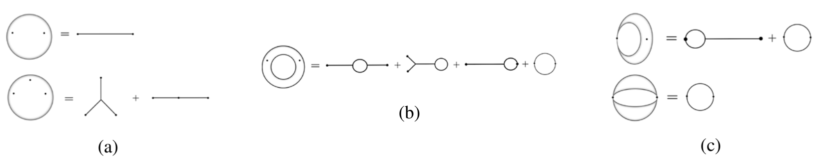

Figure 4. (a) Glueball tree graphs dual to punctured spheres in the effective theory of glueballs. (b) Glueball one-loop graphs dual to the 2-punctured torus. (c) Glueball one-loop graphs dual to pinched tori.

new topologies contribute zero to the S matrix, since they miss at least one external glueball leg, as for the last two graphs in Fig. 5. It is an open problem – though – whether an effective action of the kind in Eq. (3) exists that also nonperturbatively captures the contribution of the entire new topological sector for the correlators. However, independently of the existence of the above effective action, for the graphs

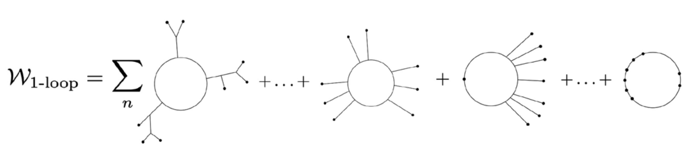

Figure 5. Glueball one-loop generating functional.

that only contain bivalent vertices, i.e. the maximally pinched ones in Fig. 3b, which are in one-to-one correspondence with diagrams only involving the subleading vertex and no leading vertex in Eq. (16), we verify the spin-statistics theorem directly by means of our asymptotic computation

$$\begin{aligned}\mathcal{W}^{E}_{\text{asym maximally pinched}}[J_{\mathbb{O}^E}, \lambda] &= -\log \mathrm{Det}\left(\mathcal{I} - \frac{1}{2}\frac{Z_{\mathbb{O}_s}(\lambda)}{\lambda^{s+2}}\Delta^{-1} P \frac{J_{\mathbb{O}^E_s}}{N} \otimes \mathcal{Y}^{E\frac{5}{2}}_{s-2}\right)\\ &= -\log \mathrm{Det}\left(\mathbb{I} - \frac{1}{2}\frac{Z_{\mathbb{O}_s}(\lambda)}{\lambda^{s+2}}\Delta^{-1} \frac{J_{\mathbb{O}^E_s}}{N} \otimes \mathcal{Y}^{E\frac{5}{2}}_{s-2}\right)\end{aligned} \tag{17}$$

thanks to the minus sign in the r.h.s. above. Finally, the occurrence of pinched tori with punctures resembles Deligne-Mumford (DM) compactification [28, 29] of the moduli space of punctured closed Riemann surfaces that arises in canonical string theories [29] because of their underlying conformal structure [29]. Yet, in general our pinched tori do not occur in the DM compactification of n-punctured tori, whose components only involve punctured closed Riemann surfaces with $\chi < 0$ [28] and, specifically, spheres with at least three punctures [28, 29] – because in the DM compactification punctures and pinches never collide [28, 29], according to the conformal structure of canonical string theories [29] – while the components of our pinched tori may contain 2-punctured spheres with $\chi = 0$. Indeed, contrary to the DM compactification [28], the dual graphs to our pinched tori may contain bivalent vertices as in the last graphs in Figs. 4c and 5. As a consequence, no canonical closed-string theory that admits the DM compactification may contain the new topological sector.

By summarizing, in large-N SU(N) YM theory a new topological sector exists that refines 't Hooft topological expansion for the correlators of twist-2 operators, both perturbatively and nonperturbatively. The new topologies solve the sign puzzle, specifically in the maximally pinched sector. They also dominate the UV asymptotics of the correlators of twist-2 operators, but contribute zero to the nonperturbative S matrix, since nonperturbatively they consist of tori with at least one pinch, corresponding to glueball one-loop diagrams with at least one missing glueball external leg. As in general the new topologies do not arise from the DM compactification of the moduli space of punctured tori, no canonical string theory admitting it may exist for the correlators of large-N YM theory in the new topological sector, but it may exist for the S-matrix amplitudes. Finally, the existence of the new topological sector – specifically, the maximally pinched one – that contributes zero to the S matrix, but nontrivially to the correlators, opens the way for a nonperturbative solution limited to the new sector by a topological field/string theory – noncanonical in the sense of the present paper – along the lines foreseen in [9], by reinterpreting the coupling to D-branes in [9] as the generating functional of correlators in the new sector [2], rather than of the collinear S-matrix amplitudes.

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