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# Feynman integrals: Synergies between particle physics and gravitational waves

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**Abstract.** Feynman integrals are essential for computing scattering amplitudes. Linear relations among these integrals, through Integral-By-Parts (IBP) identities, reduce them to a smaller set of independent integrals, known as master integrals (MIs). In twisted de-Rham cohomology, Feynman integrals form a vector space with an inner product, called the intersection number, which simplifies this reduction process. These methods have been applied in particle physics and recently extended to gravitational wave physics, notably in modeling binary black hole mergers. This proceedings highlights the synergy between these fields, showcasing how advanced techniques from Feynman integrals enable high-precision results in both areas.

## 1 Introduction

Feynman integrals play a fundamental role in the computation of scattering amplitudes, which encode all the physical information of a scattering process. As perturbation theory advances to higher orders, involving multiple loops, the complexity of these integrals increases significantly, necessitating efficient techniques to manage the vast number of integrals. In typical scattering amplitude evaluations, there can be on the order of $O(10^5)$ Feynman integrals, many of which are not independent. These integrals can be systematically reduced using Integration-By-Parts (IBP) identities [1], which reduce the large set of integrals to a smaller set of fundamental integrals, known as Master Integrals (MIs). The core idea behind IBP identities is that total derivatives with respect to loop momenta vanish in dimensional regularization, leading to a system of algebraic relations among integrals. This system can then be solved to express a large number of integrals in terms of a finite set of MIs.

Recently, it has been recognized that Feynman integrals exhibit an underlying vector space structure, particularly within parametric representations such as the Baikov or Feynman-Schwinger parametrizations. This structure is framed in terms of intersection theory within twisted de Rham cohomology [2–4]. In this framework, a bilinear pairing between differential forms, known as intersection numbers [5, 6], defines a scalar product on the vector space of Feynman integrals. This scalar product enables the projection of any multi-loop integral onto a basis of master integrals. In this context, the evaluation of intersection numbers is

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a key step, and various efficient methods have been proposed to compute these numbers [4, 7–13].

In recent years, methods used for Feynman integral computations have found surprising and impactful applications in the study of gravitational waves [14–24]. Gravitational wave phenomena, particularly from astrophysical sources such as binary black hole mergers, require precise theoretical models to compare with observational data. See [25, 26] for reviews. Remarkably, the same mathematical techniques developed for particle physics—such as IBP reduction, master integral evaluation, and dimensional regularization—have proven useful in calculating the conservative potential [27, 28], energy flux [29, 30], and waveforms [31, 32].

In this context, we have focused on the Effective Field Theory (EFT) techniques developed in [14, 27] to carry out post-Newtonian analysis in the conservative sector of the two-body problem, using an extension of the diagrammatic approach introduced in [17].

In this proceedings, we first discuss the decomposition of Feynman integrals using intersection theory, and then explore how the techniques for evaluation of Feynman integrals are applied to compute the 2-body Hamiltonian for the binary compact objects.

## 2 Decomposition of Feynman Integrals using Intersection theory

We consider an integral of the following form:

$$I = \int_C u(\mathbf{z})\, \varphi(\mathbf{z}) \ , \tag{1}$$

where $u(\mathbf{z})$ is a multi-valued function, and $\varphi(\mathbf{z}) = \hat{\varphi}(\mathbf{z})\, d^n\mathbf{z}$ represents a single-valued differential $n$-form. We impose the condition that $u(\mathbf{z})$ vanishes at the boundaries of the integration contour $C$, i.e., $u(\partial C) = 0$. This ensures the vanishing of surface terms. Assuming that $u(\mathbf{z})$ regulates all boundaries, we apply Stokes' theorem:

$$0 = \int_C d\,(u\,\xi) = \int_C (du \wedge \xi + u\, d\xi) = \int_C u \left(\frac{du}{u} \wedge + d\right) \xi \equiv \int_C u\, \nabla_\omega \xi, \tag{2}$$

where $\xi$ is a differential $(n-1)$ form and $\nabla_\omega \equiv d + \omega\wedge$, with $\omega \equiv d\log u = \sum_{i=1}^{n} \hat{\omega}_i dz_i$. Following this, we obtain:

$$\int_C u\, \varphi = \int_C u\, (\varphi + \nabla_\omega \xi)\,, \tag{3}$$

which allows us to define an equivalence class for $\varphi$ as: ${}_\omega\langle\varphi| : \ \varphi \sim \varphi + \nabla_\omega \xi$. In other words, this leads to consider the following

$$H^n_\omega = \{n\text{-forms } \varphi \,|\, \nabla_\omega \varphi = 0\} \,/\, \{\nabla_\omega \xi\}. \tag{4}$$

This is known as the *twisted cohomology group* and its elements are called *twisted cocycles*. The integration domain $C$, along with the integrand defined on its branch, forms what is called the *twisted cycle*. One can show that there exists an equivalence class of twisted cycles, which form the *twisted homology group*. Within this framework, the integral $I$ can then be interpreted as the pairing between a twisted cycle and a twisted co-cycle:

$$I = \int_C u(\mathbf{z})\, \varphi(\mathbf{z}) \equiv \langle\varphi|C]. \tag{5}$$

Then, Eq. 4 can be considered as the *vector space* of Feynman integrals. Additionally, we introduce a dual space of twisted co-cycles, the so called *dual twisted cohomology group*: $(H^n_\omega)^* = H^n_{-\omega}$ where $\nabla_\omega$ is replaced by $\nabla_{-\omega}$.

We denote the *dimension* of the (dual) twisted cohomology group by $\nu$, which is the *number* of MIs; thanks to complex Morse (Picard-Lefschetz) theory, $\nu$ is determined as the *number of critical points* of the function $\log u(\mathbf{z})$ [4, 33]

$$\nu = \dim\left(H^n_{\pm\omega}\right) = \text{number of solutions of:} \quad \hat{\omega}_i = \partial_{z_i} \log u(\mathbf{z}) = 0 \qquad 1 \le i \le n. \tag{6}$$

We consider the basis of forms, denoted as $|e_i\rangle$ for $i = 1, 2, \ldots, \nu$ and basis of the dual forms $|h_i\rangle$ to decompose an arbitrary twisted co-cycle $\langle\varphi|$. Using these twisted co-cycles and their duals, we introduce a bilinear non-degenerate pairing $\langle e_i|h_j\rangle$, known as the "intersection number." This leads to the construction of the metric matrix $\mathbf{C} = \langle e_i|h_j\rangle$, whose entries are the intersection numbers between basis forms from the twisted cohomology group and it's dual. Following [2], we obtain the master decomposition formula:

$$\langle\varphi| = \sum_{i,j=1}^{\nu} \langle\varphi|h_j\rangle \left(\mathbf{C}^{-1}\right)_{ji} \langle e_i|, \tag{7}$$

which projects $\langle\varphi|$ onto a basis of twisted co-cycles $\langle e_i|$. The integral $I$ can then be expressed as:

$$I = \int_C u\,\varphi = \sum_{i,j=1}^{\nu} \langle\varphi|h_j\rangle \left(\mathbf{C}^{-1}\right)_{ji} \int_C u\, e_i = \sum_{i,j=1}^{\nu} \langle\varphi|h_j\rangle \left(\mathbf{C}^{-1}\right)_{ji} J_i = \sum_{i}^{\nu} c_i J_i, \tag{8}$$

where $J_i$ are the independent Feynman integrals, known as master integrals (MIs) and $c_i$ are the rational coefficient of these MIs.

Following [2–4, 6–8, 34], the intersection numbers can be defined as

$$\langle e_i|h_j\rangle = \frac{1}{(2\pi i)^n} \int_X e_{i,c} \wedge h_j, \tag{9}$$

where $e_{i,c}$ is the twisted form with compact support. This integral localizes on the poles of the connection $\omega$ and can be expressed as sum of residues, employing Cauchy's formula. These methods have led to improvements in the reduction of Feynman integrals [11–13, 35, 36]. Recently, they have been applied to complete the reduction of two-loop five-point integrals [37], using polynomial reduction and companion tensor algebra.

## 3 Feynman Integrals and Post-Newtonian approximation

In the context of a bound system of two compact objects, we encounter three distinct length scales: the Schwarzschild radius $R_s$, associated with each compact object; the orbital radius $r$, describing the distance between the two objects; and the wavelength $\lambda$ of the emitted gravitational waves. Assuming that the velocities of the objects are small compared to the speed of light and the objects are far apart, the system can be treated as propagating on a flat background, i.e., $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where the gravitational interaction between the two objects is mediated by the graviton field $h_{\mu\nu}$. This scenario leads to the following hierarchy of length scales: $\lambda \gg r \gg R_s$. Our primary interest is in the long-distance physics at the wavelength scale $\lambda$. To simplify the analysis, we decompose the graviton field into two distinct components: short-distance potential gravitons $H_{\mu\nu}$, which scale as $(k_0, \mathbf{k}) \sim (v/r, 1/r)$, and long-distance radiation gravitons $\bar{h}_{\mu\nu}$, scaling as $(k_0, \mathbf{k}) \sim (v/r, v/r)$[14]. Using the virial theorem (or equivalently, Kepler's third law), we have $v^2 \sim 1/r$ for bound orbits. Thus, the dimensionless expansion parameter can be expressed as $v^2/c^2 \sim G_N M/rc^2$, which formally scales as $1/c^2$. Consequently, following standard post-Newtonian (PN) literature, we adopt a formal expansion in powers of $1/c$, with one PN order corresponding to $1/c^2$.

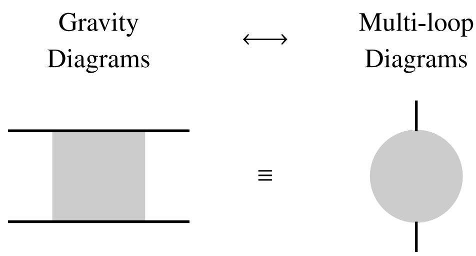


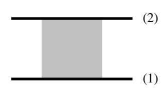

**Figure 1.** The diagrammatic correspondence between the four-point EFT-Gravity graphs and the two-point quantum-field-theory (QFT) graphs.

To compute the conservative binding potential of the two-body system, we neglect the radiation modes and focus on the potential modes using the Kaluza-Klein (KK) decomposition [38, 39]. In this decomposition, the components of the metric $g_{\mu\nu} = \eta_{\mu\nu} + H_{\mu\nu}$ are parameterized in terms of three fields: a scalar field $\boldsymbol{\phi}$, a 3-dimensional vector field $\boldsymbol{A}_i$, and a 3-dimensional symmetric rank-2 tensor field $\boldsymbol{\sigma}_{ij}$. The effective action for the two-body system is obtained by integrating out the gravitational degrees of freedom. This is expressed as:

$$\exp\left[\mathbf{i}\int dt\,\mathcal{L}_{\text{eff}}\right] = \int D\boldsymbol{\phi} D\boldsymbol{A}_i D\boldsymbol{\sigma}_{ij}\, e^{\mathbf{i}(S_{\text{EH}}+S_{\text{pp}})}\,, \tag{10}$$

where, $S_{\text{EH}}$ is the Einstein-Hilbert action describing the gravitational field and $S_{\text{pp}}$ is the worldline point particle action [40]. $\mathcal{L}_{\text{eff}}$ is the effective Lagrangian, further decomposed into kinetic and potential terms as: $\mathcal{L}_{\text{eff}} = \mathcal{K}_{\text{eff}} - \mathcal{V}_{\text{eff}}$, with $\mathcal{K}_{\text{eff}}$ being the kinetic term and $\mathcal{V}_{\text{eff}}$ representing the effective potential due to gravitational interactions. The effective potential $\mathcal{V}_{\text{eff}}$ can be computed from connected, classical, 1-particle irreducible (1PI) scattering amplitudes as follows:

$$\mathcal{V}_{\text{eff}} = \mathbf{i}\lim_{d\to 3}\int_{\mathbf{p}} e^{\mathbf{i}\mathbf{p}\cdot(\mathbf{x}_{(1)}-\mathbf{x}_{(2)})}$$




, (11)

where $\mathbf{p}$ represents the momentum transfer between the two particles, and the box diagram refers to all possible Feynman diagrams with gravitons $\boldsymbol{\phi}, \boldsymbol{A}_i, \boldsymbol{\sigma}_{ij}$ mediating the gravitational interaction between the two point particles, depicted by the two solid black lines. To obtain the effective potential from the diagrammatic approach as shown in equation (11), we begin by generating all the relevant generic topologies contributing at different orders of $G_N$. Within the EFT framework, the sources remain static and as a result the generated Feynman diagrams are mapped to two-point multi-loop Feynman diagrams with mass-less internal lines and an external momentum (the momentum transferred between two sources) as shown in figure 1. We translate these Feynman diagrams to their corresponding Feynman amplitudes after performing the tensor algebra using by means of an in-house code that uses tools from `EFTofPNG` [41] and `xTensor` [42].

We perform the reduction of the multi-loop tensor integrals to scalar integrals by applying a set of projectors exploiting the Lorentz invariance. The generated scalar integrals are reduced by using the `LiteRED` [43] to the MIs. After the evaluation of the MIs, we perform the Fourier transform to obtain the effective potential by expanding the complete scattering amplitude as Laurent series in $\epsilon$ around $d = 3$.

The computation of the effective potential starting from the generation of the required Feynman diagrams, expressing them in multi-loop integrands, performing IBP reduction, and then applying the Fourier transformations have been automated through an in-house code, elaborating on some of the ideas implemented in `EFTofPNG` [41], and using `xTensor` [42] for tensor algebra manipulations as well as successful interface to `LiteRed` [43] for the IBP reduction. Employing this framework, we computed the complete evaluation of $N^3$LO Post-Newtonian (PN) correction to the the spin-orbit Hamiltonian [44], quadratic-in-spin Hamiltonian [45] for rapidly rotating compact objects, within the effective field theory diagrammatic approach of General Relativity. Additionally, we derived an effective Hamiltonian that describes the dynamical gravitoelectric tidal interaction between two nonspinning compact objects up to the 3PN order [46, 47].

## 4 Conclusion

Thus, the study of Feynman integrals forms a crucial bridge between particle physics and gravitational wave research. By leveraging the synergies between these fields, we not only improve our understanding of the fundamental forces of nature but also advance our ability to detect and interpret the ripples in spacetime caused by massive astrophysical events. This proceedings discusses the recently found connections, focusing on the computational techniques that make these breakthroughs possible and highlighting the ongoing interplay between Feynman integral and gravitational wave physics.

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