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On the equation for the vertical velocity of atmospheric air

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Abstract. The conditions for the inertialess approximation of the equation for the vertical velocity of humid atmospheric air are considered, under which the inertia of the vertical and horizontal motion can be considered small. The resulting equation can be used in calculations of atmospheric processes.

1 Introduction

R.I. Nigmatulin considered in his work [1] the atmospheric circulation equations, in which the equation of the vertical momentum component is usually considered in the inertialess approximation. At the same time, based on the non-stationary continuity equation and the balance of thermal energy, the equation for the vertical air velocity w (which, generally speaking, is not quasi-static due to the influence of horizontal accelerations) was considered

$$\frac{\partial w}{\partial z} = -div_H(\boldsymbol{v}) - \frac{\dot{M}}{\gamma M} + \frac{(\gamma-1)Q'}{\gamma g M} - \frac{[\vec{v}\nabla(p)]_H}{\gamma p}, \tag{1}$$

where γ is the adiabatic exponent for air, g is the acceleration of gravity.

$$M = \int_z^{H(t)} \rho dz\,, \dot{M} = -\int_z^{H(t)} \left(\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y}\right) dz \tag{2}$$

$$div_H(\boldsymbol{v}) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}, \tag{3}$$

$$[\boldsymbol{v}\nabla(p)]_H = u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y}, \tag{4}$$

where Q' is the total volumetric heat influx into the atmosphere due to radiation, turbulent heat conduction, radiation absorption, *u, v* are the horizontal components of the wind speed. H(t) is the height of the postulated "upper boundary of the atmosphere", the pressure on which corresponds to the tabular value and is the boundary condition that ensures the applicability of the continuous medium model. Based on the analysis of the characteristic values of atmospheric parameters in [1], it was concluded that (4) in Eq. (1) is small, which is confirmed in this work. Therefore, in [1], equation (1) was presented in the form

$$\frac{\partial w}{\partial z} = -div_H(\vec{v}) - \frac{\dot{M}}{\gamma M} + \frac{(\gamma-1)Q'}{\gamma g M} \tag{5}$$

In the future, the comparison of terms (3) and (4) in equation (1) was carried out for the “free atmosphere” region and for the surface region based on reasoning similar to the proof of the Taylor-Proudman theorem [2]. In addition, additional conditions are considered under

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which, in the relation for the "horizontal" part of the velocity divergence in (1) (that is, the mass flow carried by the wind), the assumption of the smallness of horizontal accelerations $\boldsymbol{a}$ is applicable. Indeed, in the most famous works of A.S. Monin and A.M. Yaglom [3], as well as S.S. Zilitinkevich [4], inertialess approaches to modeling the surface layer are considered, therefore, refinement of the conditions for applicability of the assumption of quasi-stationarity of the model equations [1] up to date.

2 Quasi-static model

Acting by analogy with [5], we introduce a local coordinate system with the 'x'-axis directed (e_1) along the Earth's parallel to the east, the 'y'-axis directed (e_2) along the meridian to the north and the 'z' axis directed (e_3) up along the local vertical. Let us write the equation of motion in the form of the Taylor-Ekman model, where the kinematic turbulent viscosity ν^{τ}, in accordance with the Boussinesq hypothesis, is presented in the same way as in the Reynolds equations [6].

$$\Omega_x = 0, \Omega_y = \Omega\, cos(\varphi)\,, \Omega_z = \Omega\, sin(\varphi)\,, a = e_1 a_x + e_2 a_y + e_3 a_z,$$
$$v = e_1 v + e_2 v + e_3 w, a + 2[\Omega \times v] = -\frac{1}{\rho}\nabla p - g + v^{\tau} e_1 \frac{\partial^2 u}{\partial z^2} + v^{\tau} e_2 \frac{\partial^2 u}{\partial z^2} \tag{6}$$

Here φ-angle of geographic latitude, Ω- angular velocity of the Earth's rotation, a - acceleration, v - speed with components (u, v, w) along the coordinate axes, $\rho=\rho(T,p,\kappa_{\text{v}})$ - corresponds to a homogeneous mixture of perfect gases ; dry air and water vapor, T-temperature, p-pressure, k_{v} - mass fraction of water vapor. ν^{τ} - vertical coefficient of turbulent transport.

In accordance with [1], for the further specification of the derivative of atmospheric pressure with respect to time $\partial p/\partial t$, we integrate the vertical component (6) over the z coordinate and obtain in an approximation that is inertialess along the vertical.

$$\left(for\ \varepsilon = \left| \frac{(2\Omega cos(\varphi) - a_z)}{g} \right| \ll 1 \right)$$
$$p = g \int_z^{H(t)} \rho dz$$
$$\frac{\partial p}{\partial t} = g \int_z^{H(t)} \frac{\partial \rho}{\partial t} dz + g \left(\rho \frac{dH}{dt} \right)_{z=H(t)},$$

whence, taking into account the form of the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0$$

we obtain from the relation for $\frac{\partial p}{\partial t}$

$$\frac{\partial p}{\partial t} = g\dot{M} - g \int_z^{H(t)} \frac{\partial \rho w}{\partial z} dz + g \left(\rho \frac{dH}{dt} \right)_{z=H(t)} =$$
$$= g\dot{M} + g\left[(\rho w) - (\rho w)_{z=H(t)} \right] + g \left(\rho \frac{dH}{dt} \right)_{z=H(t)}, \tag{7}$$

or

$$\frac{\partial p}{\partial t} = g\rho w + g\dot{M} \tag{8}$$

Let us apply the operator rot to the left and right parts of (6). As a result, similarly to how it is done in the proof of Proudman's theorem, we obtain for the "horizontal" terms of the velocity divergence

$$div_H(\vec{v}) \equiv \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = f_{v+} \left(\frac{\partial \lambda}{\partial x} u + \frac{\partial \lambda}{\partial y} v \right), \tag{9}$$

$$d\lambda = \frac{dT}{T}\left(\frac{\mu_{air}}{\mu_{vap}} - 1\right) d\kappa_v,$$

$$f_v = \frac{\nu^\tau}{2\Omega sin(\varphi)}\left[\frac{\partial^2}{\partial z^2}\left(rot(v)\right)_Z + \left(\frac{\partial^2 u}{\partial z^2}\frac{\partial \lambda}{\partial y} - \frac{\partial^2 v}{\partial z^2}\frac{\partial \lambda}{\partial x}\right)\right] + ctg(\varphi)\left[\frac{\partial w}{\partial y} - 0.5w\frac{\partial \lambda}{\partial y}\right] + \\ + \frac{1}{2\Omega sin(\varphi)}\left[-\left(rot(a)\right)_Z + \left(a_y\frac{\partial \lambda}{\partial x} - a_x\frac{\partial \lambda}{\partial y}\right)\right] \qquad (10)$$

(Here $(rot(A))_z$ is the z-component of the rotor of the vector A), where *u,v,w* are the velocity components along the basis vectors, μ_{air} and μ_{vap} are the molecular weights of air and water, respectively. In this case, the integrand $(\partial\rho u/\partial x + \partial\rho v/\partial y)$ in (2) will take the form

$$\frac{\partial \dot{M}}{\partial z} = \left(\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y}\right) = \rho\left[f_v + f_p\right] \qquad (11)$$

$$f_p = \frac{1}{p}\left(u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y}\right) \qquad (12)$$

where p is the air pressure. Next, we estimate the quantities f_v and f_p. To estimate the value of f_v on the right side of (11), we denote the characteristic values; ω_c- vertical component $rot(v)$ for large-scale vortices, ν^τ - viscosity for vertical motions, w_0 - vertical velocity, L_z - vertical scale, L_h - horizontal scale, u_h- wind speed, $a_h \sim \frac{u_h^2}{L_h}$ - horizontal acceleration. Then the scales of the terms in square brackets on the right side f_v of relation (10) will be, respectively, of the order $\tau_\Omega^{-1} = \omega_c Ek$,

$$\tau_w^{-1} = \frac{w_0}{L_h}, \qquad \tau_a^{-1} = \Omega Ro^2$$

$$Ro = \frac{u_h}{2\Omega\, sin(\varphi)\, L_h} - Rossby\ number, \qquad Ek = \frac{\nu^\tau}{2\Omega\, sin(\varphi)\, L_z^2} - Ekman\ number$$

For small horizontal scales L_h, the second and third terms can be significant, and for large ones L_h the second term decreases as L_h^{-1}, and the third as L_h^{-2}. Therefore, for large and for gradients of the function corresponding to the characteristic horizontal inhomogeneities of the atmospheric parameters according to the tables [7], the value of f_v can be represented by an approximate relation

$$f_v \approx \frac{1}{\tau_\Omega}\frac{\partial^2 \boldsymbol{rot}(\boldsymbol{v})_z}{\partial z^2} \qquad (13)$$

Based on equations (2) for the horizontal components of the momentum, we obtain on the right side of (12) for f_p

$$f_p = \frac{1}{p}\left((\nabla p)\boldsymbol{v}\right)_h = \frac{\gamma}{C^2}\left[-2\Omega\, cos(\varphi)\, u\, w - (\boldsymbol{av})_h + \nu^\tau\left(u\frac{\partial^2 u}{\partial z^2} + v\frac{\partial^2 v}{\partial z^2}\right)\right], \\ where\ (\boldsymbol{av})_h \equiv \left(a_x u + a_y v\right), \left((\nabla p)\boldsymbol{v}\right)_h = \frac{\partial p}{\partial x}u + \frac{\partial p}{\partial y}v. \qquad (14)$$

where γ is the adiabatic index for air, C^2 is the square of the speed of sound. For heights corresponding to the "free atmosphere", where the geostrophic approximation is applicable, the value $f_p$can be considered negligible, since in this approximation the wind speed is directed perpendicular to the horizontal component of the atmospheric pressure gradient. For heights corresponding to the surface atmospheric layer (Z<100 m), the third term is the main one on the right side of (14). The assessment carried out according to the data of [7] shows that for the characteristic values of wind speeds and horizontal inhomogeneity of the atmospheric temperature, the value of f_p is significantly less than the corresponding terms in f_v (relation (10)), which also contain derivatives

$$\frac{\partial^2 u}{\partial z^2}, \frac{\partial^2 v}{\partial z^2}.$$

Thus, for large synoptic characteristic scales, the left side of relation (11) and, consequently, $\dot{M}$ in (5) are quasi-static and can be considered in the approximation of negligibly small horizontal accelerations.

This statement corresponds to the observational data given in the monograph by L. T. Matveev [8]. In [8], the dependence of the characteristic scale of the vertical air velocity on the corresponding horizontal scale of the region over which vertical flows were observed was noted. For small horizontal scales, vertical velocity values can exceed meters per second. This is due to the influence of the inertia of horizontal wind flows. But large horizontal scales correspond to small average (10^{-3} -10^{-2} m/s) values of the vertical velocity.

Next, we specify the expressions for $\dot{M}, \partial\dot{M}/\partial z$ .Taking into account only the main terms on the right side of (10), we obtain

$$\frac{\partial\dot{M}}{\partial z} = \nu^{\tau}(z)\frac{\rho}{2\Omega sin(\varphi)}\left\{\frac{\partial^2 u}{\partial z^2}\frac{\partial\lambda}{\partial y} - \frac{\partial^2 v}{\partial z^2}\frac{\partial\lambda}{\partial x} + \frac{\partial^2 \boldsymbol{rot}(\boldsymbol{v})_z}{\partial z^2}\right\} \quad (15)$$

Since the direction $n = e_1\cos(\psi) + e_2\sin(\psi)$ (ψ-angle between wind direction and latitude) of the wind speed V_h (z) in the surface layer can be considered unchanged in height, we assume

$$u = V_h(z)\,cos(\psi)\,, v = V_h(z)\,sin(\psi),$$

then

$$\frac{\partial\dot{M}}{\partial z} \approx \nu^{\tau}(z)\frac{\rho}{2\Omega sin(\varphi)}\left[\frac{\partial^2 V_h}{\partial z^2}(\nabla\lambda\times n)_z + \frac{\partial^2 (rot(\boldsymbol{v}))_z}{\partial z^2}\right] \quad (16)$$

$$\dot{M} = -\int_z^{H(t)}\frac{\partial\dot{M}}{\partial z}dz.$$

To specify the right side of (16), we consider, in the approximation of the Monin-Obukhov model, the relations that model the height distribution for the turbulent transfer coefficient and wind speed. Then the dependence of the wind speed V_h on the height is determined by the relation

$$V_h(z) = \frac{v_*}{k}ln\left(\frac{\eta}{\eta_0}\right), \eta = exp\left(\frac{z}{L_*}\right) + \frac{\nu^m}{v_*L_*} - 1, \eta_0 = exp\left(\frac{z_0}{L_*}\right) + \frac{\nu^m}{v_*L_*} - 1. \quad (17)$$

where z_0 is the roughness parameter. (z_0 ~ 0.1 of the characteristic size of the surface roughness) ν_m, is the molecular kinematic viscosity of air.

$$L_* = -\frac{v_*^3 c_p\,\rho}{k\beta q} \quad (18)$$

where $v_* = \sqrt{\frac{\tau_0}{\rho}}$ - friction velocity (τ_0 is the friction stress at z=0), k =0.4 is the Karman number, β=g/*T* is the buoyancy parameter, *q* is the heat flux, positive if directed upwards from the earth's surface. Let's assume that $(rot(v))_z$ changes with height in the same way as the wind speed

$$\left(rot(v)\right)_z = \frac{\omega_*}{k}ln\left(\frac{\eta}{\eta_0}\right) \quad (19)$$

If at an altitude of about 1 km $\left(\mathrm{rot}(v)\right)_z \approx 4 10^{-5} s^{-1}$,then $|\omega_*|\sim 10^{-7}\ s^{-1}$. For small values of vorticity $|\omega_*| << 10^{-7}\ s^{-1}$, the main term in square brackets on the right side of (16) becomes the first term, determined by the inhomogeneity of temperature and humidity distribution in the horizontal direction. The turbulent transport coefficient ν^τ in accordance with [8] has the form of the Izvekov formula

$$\nu^{\tau}(z) = \nu_{\infty}^{\tau}\left[1 - (1-\varepsilon)\,exp\left(-\frac{z}{L_*}\right)\right]. \varepsilon = \frac{\nu^m}{\kappa v_* L_*}, \nu_{\infty}^{\tau} = \kappa v_* L_*, \quad (20)$$

After substituting the relations for V_h and ν^τ into (16), we obtain

$$\frac{\partial\dot{M}}{\partial z} = \nu^{\tau}(z)\frac{\rho}{2\Omega\,sin(\varphi)}\left[\frac{\partial^2 V_h}{\partial z^2}(\nabla\lambda\times\boldsymbol{n})_z + \frac{\partial^2 rot(\boldsymbol{v})_z}{\partial z^2}\right] =$$

$$\frac{\rho}{2\Omega\,sin(\varphi)}\frac{v_*[v_*(\nabla\lambda\times\boldsymbol{n})_z+\omega_*]}{L_*}\left(\frac{\nu^m}{v_*L_*} - 1\right)\frac{1}{\eta(z)} \quad (21)$$

When evaporating from the ocean surface, the quantity *q* included in L_* is negative. Therefore L_* >0 and the value ν^τ is limited. In the future, it will be taken equal to the average height value $<\nu^\tau>$.

$$\dot{M} = \langle \nu^{\tau} \rangle \frac{\rho[v_* (\nabla\lambda \times \boldsymbol{n})_z + \omega_*]}{2\Omega sin(\varphi)} \left(1 - \frac{\nu^m}{v_* L_*}\right) \left[\frac{1}{\eta(z)} - \frac{1}{\eta(H(t))}\right] \tag{22}$$

Taking into account (11), we rewrite the continuity equation in the form

$$\frac{\partial \rho w}{\partial z} = -\frac{\partial \dot{M}}{\partial z} - \frac{\partial \rho}{\partial t}$$

Because the $\rho=\rho(T,p,\kappa_v)$,

$$\frac{\partial \rho}{\partial p} = \frac{\rho}{p}, \quad \frac{\partial \rho}{\partial T} = -\frac{\rho}{T}, \quad \frac{\partial \rho}{\partial \kappa_v} \approx \rho\left(\frac{\mu_a}{\mu_v} - 1\right),$$

we get finally

$$\frac{\partial \rho w}{\partial z} + \frac{\rho w}{L_g} = -\left(\frac{\dot{M}}{L_g} + \frac{\partial \dot{M}}{\partial z}\right) + \rho\left[\frac{1}{T}\frac{\partial T}{\partial t} - \left(\frac{\mu_a}{\mu_v} - 1\right)\frac{\partial \kappa_v}{\partial t}\right], \tag{23}$$

$$L_g = C^2/(\gamma g)$$

Note that in (23) the non-stationary terms on the right-hand side are due only to heat transfer and diffusion processes.

3 Conclusion

The equation for the vertical velocity, proposed in the work of R.I. Nigmatulin [1], can be considered inertial in all spatial variables for sufficiently large (synoptic) characteristic horizontal scales of modeling.

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