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Spectrum of Reynolds stress for anisotropic turbulence and Taylor expansion of small-scale velocity

Alexander Balonishnikov*

Department of Mathematics and Mechanics, Department of Physics, Saint-Petersburg State University of Aerospace Instrumentation (SUAI), 67, Bolshaya Morskaia str., SPb, 190000, Russia

Abstract. Solution of the small-scale velocity transport equation for incompressible fluid in anisotropic turbulence as Taylor series in time is used for obtaining of explicit expression for the Reynolds Stress spectra. The initial condition is isotropic with a given turbulent energy spectrum. It is assumed that low-order terms are sufficient for correct description for the period about the time turnover of large-scale eddies. The averaging is taken over this period, the integral scale of turbulence and three uniformly distributed random phases of initial isotropic velocity. It is assumed that isotropic energy spectrum is given. The Craya-Herring components of small-scale velocity (polarization Fourier components) are used. Obtained Reynolds stress contain the mean dissipation rate of kinetic energy.

1 Introduction

The problem of turbulence remains one of the most important unsolved problems in classical physics. Most researchers believe that in the incompressibility approximation of the fluid, the initial equations are the Navier -Stokes equations. In recent years, doubts have arisen that this is true for flow of real gases. Molecular thermal fluctuation can significantly blur the characteristic cascade picture of energy transition from large eddies to increasingly smaller ones and energy dissipation into heat [1]. At least for liquid flows, such doubts have not yet arisen and we will assume that e are dealing with exactly this case. A further important approximation (not valid for all types of turbulence) is that turbulent flow is homogeneous but anisotropic. That is the large-scale part of the flow changes relatively little in space and time, and the small-scale part has rich dynamics, including the Richardson-Kolmogorov cascade process of energy transfer from large eddies to small ones. The closure problem is to find a relation between the Reynolds stresses and the components of the large-scale velocity gradient tensor. Off-diagonal components of Reynolds stresses in a certain range of scales have a Lumley spectrum [2]. The monograph [3] is devoted to the anisotropic homogeneous turbulence of an incompressible fluid. In most cases, even such a simplification requires the use of numerical methods for integrating three- dimensional turbulent flows; a typical example is work [4].

* Corresponding author: balonishnikov@guap.ru

It is proposed to use a simpler equation for the transport of small-scale velocity components than the one usually used [3]. The difference is that the large-scale velocity in our approach is considered a constant value, and not linear as is generally accepted. Of course, mean velocity gradients are present in the small-scale velocity transport equations in both approaches. The advantage of our approach is that ordinary Fourier transform can be used for analysis, as well as for analysis of isotropic turbulence. It is especially convenient to use two polarization components (Craya-Herring decomposition [5,6]) instead of three conventional components. The analysis of dynamics in a first approximation, neglecting nonlinearities, was previously described in our earlier work. In a subsequent study, we took nonlinearity into account using a 'naïve' approximation approach [7]. Here we obtain expressions for the Reynolds stresses spectra using the first terms of Taylor expansion in time series. Our approach differs from the rapid distortion theory [8]. It is also important to mention two important recent reviews devoted to analytical theories of turbulence [9,10].

2 Craya-Herring decomposition

Following [11], we introduce the wave vector $k = (k_x . k_y, k_z)$. Incompressibility equation for conventional Fourier components of small-scale velocity has the form:

$$k \cdot v(k,t) = 0 \tag{1}$$

The transition to the polarization components of small-scale velocity is carried out by introducing four unit vectors:

$$\mathrm{e}^{1}(\mathrm{k}) = \frac{\mathrm{k}\times \mathrm{e}_4}{|\mathrm{k}\times \mathrm{e}_4|} = \frac{1}{\sqrt{{\mathrm{k_x}}^2+{\mathrm{k_y}}^2}}(\mathrm{k_y}; -\mathrm{k_x}; 0) = (\sin\phi\,; -\cos\phi\,; 0)$$

$$\mathrm{e}^{2}(\mathrm{k}) = \frac{\mathrm{k}\times(\mathrm{k}\times \mathrm{e}_4)}{|\mathrm{k}\times(\mathrm{k}\times \mathrm{e}_4)|} = \frac{1}{\sqrt{\mathrm{k}_x^2+\mathrm{k}_y^2+\mathrm{k}_z^2}\sqrt{\mathrm{k}_x^2+\mathrm{k}_y^2}}(\mathrm{k_z k_x}; \mathrm{k_z k_y}; -\mathrm{k}_x^2-\mathrm{k}_y^2)$$

$$= (\cos\vartheta\cos\phi\,; \cos\vartheta\sin\phi\,; -\sin\vartheta)$$

$$\mathrm{e}_3(\mathrm{k}) = \frac{\mathrm{k}}{|\mathrm{k}|} = \frac{1}{\sqrt{{\mathrm{k_x}}^2+{\mathrm{k_y}}^2+{\mathrm{k_z}}^2}}(\mathrm{k_x}; \mathrm{k_y}; \mathrm{k_z}) = (\sin\vartheta\cos\phi\,; \sin\vartheta\sin\phi\,; \cos\vartheta)$$

$$e_4(k) = (0; 0; 1),$$

Where angular variables $\phi, \quad \vartheta$ change at intervals

$$0 \le \phi \prec 2\pi, \ \ 0 \le \vartheta \le \pi.$$

The transport equation for small-scale polarization Fourier velocity components will have the next form in a coordinate system moving with velocity $V(x,t)$:

$$(\partial_t + \nu k^2)u^\gamma(k,t) = a^{\gamma\mu}u^\mu + \sum p,q,p+q\Phi^{\gamma\alpha\beta}(k,p,q)u^\alpha(p,t)u^\beta(q,t) \tag{2}$$

where

$$a^{\gamma\mu} = -e_j^\gamma e_m^\mu \partial_m V_j, \ \Phi^{\gamma\alpha\beta}(k,p,q) = -ik_m e_j^\gamma(k) e_j^\alpha(p) e_m^\beta(q);$$

here the upper indices take the values 1,2 and the lower indices take the values 1,2,3, and also summation is implied over the repeating indices. There is the following connection between polarization and ordinary Fourier components of velocity $u^\alpha(k,t)$ and $v_j(k,t)$: $v_j = e_j{}^\alpha \ u^\alpha$; $u^\alpha = v_j e_j^\alpha$. There are also useful identities $e_j^\alpha e_l^\alpha = P_{jl} = \delta_{jl} - k_j k_l/k^2$, $k \times e^\alpha = 0$, $e^\alpha \cdot e^\beta = \delta_{\alpha\beta}$, where Greek indices run through the values 1 and 2, and Latin 1,2,3.

Let's consider the case of a simple shear: the velocity gradient tensor has only one component that is nonzero: $S = \partial_1 V_2 \ne 0$.

Then

$$a^{11} = \frac{1}{2} sin\, 2\, \phi \partial_1 V_2$$
$$a^{12} = cos^2\, \phi\, cos\, \vartheta\, \partial_1 V_2$$
$$a^{21} = -\, cos\, \vartheta\, sin^2\, \phi\, \partial_1 V_2$$
$$a^{22} = -\frac{1}{2} cos^2\, \vartheta\, sin\, 2\, \phi \partial_1 V_2$$
$$\Phi^{111}(k,p,q) = -ik\, sin\, \vartheta_k\, sin(\phi_q - \phi_k)\, cos(\phi_k - \phi_p)$$
$$\Phi^{112} = -ik[sin\, \vartheta_k\, cos\, \vartheta_q\, sin(\phi_q - \phi_k) - sin\, \vartheta_q\, cos\, \vartheta_k]\, cos(\phi_k - \phi_p)$$
$$\Phi^{121} = -ik\, sin\, \vartheta_k\, sin(\phi_q - \phi_k)\, cos\, \vartheta_p\, sin(\phi_k - \phi_p)$$
$$\Phi^{122} = -ik[sin\, \vartheta_k\, cos\, \vartheta_q\, cos(\phi_k - \phi_q) - sin\, \vartheta_q\, cos\, \vartheta_k]\, cos\, \vartheta_p\, sin(\phi_k - \phi_p)$$
$$\Phi^{211} = -ik\, sin\, \vartheta_k\, sin(\phi_q - \phi_k)\, cos\, \vartheta_k\, sin(\phi_k - \phi_p)\, sin(\phi_p - \phi_k)$$
$$\Phi^{212} = -ik[sin\, \vartheta_k\, cos\, \vartheta_q\, cos(\phi_k - \phi_q) - cos\, \vartheta_k\, sin\, \vartheta_q]\, cos\, \vartheta_k\, sin(\phi_p - \phi_k)$$
$$\Phi^{221} = -ik\, sin\, \vartheta_k\, sin(\phi_q - \phi_k)[cos\, \vartheta_k\, cos\, \vartheta_p\, cos(\phi_k - \phi_p)\, sin\, \vartheta_k\, sin\, \vartheta_p$$
$$\Phi^{222} = -ik[sin\, \vartheta_k\, cos\, \vartheta_q\, cos(\phi_k - \phi_q) - cos\, \vartheta_k\, sin\, \vartheta_q][cos\, \vartheta_k\, cos\, \vartheta_p\, cos(\phi_k - \phi_p) + sin\, \vartheta_k\, sin\, \vartheta_p]$$

Initial condition for equation (2) is isotropic state:

$$< v_\alpha(k) v_\beta(-k) >= \frac{2P_{\alpha\beta}(k)E(k)}{(d-1)A_d k^{d-1}}$$

where $A_d = 2((\pi)^{d/2}/\Gamma(d/2))$ - is the square of unit sphere surface in d-dimensional space, $E(k) -$ is the given turbulent energy spectrum [12]. Let's chose, following [13], energy spectrum $E(k)$ as:

$$E(k) = K_0 \varepsilon^{2/3} k^{-5/3} f(k\eta),$$
$$f(k\eta) = exp(-1.5K_0(k\eta)^{4/3})$$

where $K_0 = 1.4 - 1.8$ is – Kolmogorov constant, $\eta = \nu^{3/4}/\varepsilon^{1/4}$- Kolmogorov microscale, ν - molecular viscosity, ε - mean dissipation rate in physical space.

Here we have $d = 3$, $A_3 = 4\pi$ and $< v_\alpha(k) v_\beta(-k) >=< v_\alpha(k) v_\beta(k)^* >= \frac{P_{\alpha\beta}(k)E(k)}{4\pi k^2}$

For polarization Fourier components of velocity, this equality takes the form:

$$e_j^\alpha e_m^\beta < u^\alpha u^{\beta *} >= \frac{P_{jm}(k)E(k)}{4\pi k^2}$$

Taking the trace of this expression using lower Latin indices and taking into account that e^1 and e^2 are orthonormal, we obtain the following expression:

$$|u^1|^2 + |u^2|^2 = \frac{E(k)}{2\pi k^2}.$$

For isotropic turbulence, it is logical to assume that

$$|u^1| = |u^2| = \frac{1}{2k}\sqrt{\frac{E(k)}{\pi}}.$$

Following [14,15]:

$$u^1(0) = \frac{1}{2k}\sqrt{\frac{E(k)}{\pi}} e^{i\chi_1} cos\, \zeta \quad (3)$$

$$u^2(0) = \frac{1}{2k}\sqrt{\frac{E(k)}{\pi}} e^{i\chi_2} sin\, \varsigma \quad (4)$$

where χ_1, χ_2, ς - are random numbers, uniformly distributed over the interval $(0,2\pi)$.

For an approximate determination of changes in small-scale velocities $u^1(t)$ and $u^2(t)$ at $0 \le t \le T = 1/S$we neglect nonlinear and viscous terms in the equation (2). Since the period of averaging of Reynolds stresses over time is relatively small, we will not use the solution here in the form $exp(\lambda_i t)$, where λ_1 and λ_2 - are eigenvalues of linear problem. Instead, we will use the first terms of the Taylor series to determine $u^1(t)$ and $u^2(t)$:

$$u^1(t) = u^1(0)(1 + a^{11}t) + a^{12}u^2(0)t \quad (5)$$
$$u^2(t) = u^2(0)(1 + a^{22}t) + a^{21}u^1(0)t \quad (6)$$

The transition to ordinary Fourier components is carried out according to the formulas:

$$v_1 = e_1^1 u^1 + e_1^2 u^2 \; ; \; v_2 = e_1^2 u^1 + e_2^2 u^2 \; ; \; v_3 = e_3^1 u^1 + e_3^2 u^2 \qquad (7)$$

Averaging over three random phases and time T for any function g is defined in the standard way:

$$< g >= \frac{1}{(2\pi)^3 T} = \int_0^{2\pi} d\varsigma \int_0^{2\pi} d\chi_1 \int_0^{2\pi} d\chi_2 \int_0^T g dt$$

As a result, we obtain spectra, expressed in term of the polarization velocity components:

$$< u^1 u^{1*} >= \frac{E(k)}{8\pi k^2} [1 + 2a^{11}T + (a^{11})^2 \frac{T^2}{3} + (a^{12})^2 \frac{T^2}{3}]$$

$$< u^2 u^{2*} >= \frac{E(k)}{8\pi k^2} [1 + 2a^{22}T + (a^{22})^2 \frac{T^2}{3} + (a^{21})^2 \frac{T^2}{3}]$$

$$< u^1 u^{2*} >= \frac{E(k)}{8\pi k^2} [(a^{21} + a^{12}) \frac{T}{2} + (a^{11}a^{21} + a^{12}a^{22}) \frac{T^2}{3}]$$

Next, using formulas (7), we find the stress spectra expressed in term of the usual Fourier velocity components. Since the main contribution to the Reynolds stresse comes from the eddies from inertial interval, we use the Kolmogorov spectrum for further calculation:

$$E(k) = Ko\varepsilon^{2/3} k^{-5/3}.$$

To obtain 9 components of the Reynolds tensor, we integrate the obtained expressions in the spherical coordinate system. Modulo radius integration gives the following expressions: $\int_{p_0}^{\infty} E(k) dk = \frac{3Ko}{2} \varepsilon S^{-1}$, because from the dimensional considerations $p_0 = S^{3/2} \varepsilon^{-1/2}$.

The result of the calculation is written in the following form:

$$< v_1 v_3 >= 0$$

$$< v_2 v_3 >= 0$$

$$< v_1 v_2 >= -\frac{3 * 1.4 Ko\varepsilon}{16S}$$

$$< v_1^2 >= \frac{3 * 2.08 Ko\varepsilon}{16S}$$

$$< v_2^2 >= \frac{3 * 2.8 Ko\varepsilon}{16S}$$

$$< v_3^2 >= \frac{3 * 1.5 Ko\varepsilon}{16S}.$$

As it can be expected, considering nonlinear terms may improve the obtained estimates.

3 Conclusion

Despite the fact that the obtained expressions contain the mean dissipation rate, they allow one to find the ratio of the Reynolds tensor components, that is, to evaluate their relative contributions. We can try to apply the resulting approach to derive the dissipation transport equation. In this case, the nonlinearity of the process and the structure of the energy spectrum in dissipation range are important. It should be expected that each type of spectrum will lead to its own dissipation transport equation.

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