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The research of the formation of phase portraits of linear second-order systems taking into account the dynamic characteristics of the model

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Abstract. The article discusses the phase space method, widely used in physics and mathematics, from the perspective of studying dynamic systems. A second-order linear system is considered, a vectorized model of the analog equivalent of the system is constructed. A system interface has been developed that allows changing the model coefficients in real time, which simulates the process of generating disturbing influences. The equilibrium positions of the system are determined, such as "stable focus", "unstable focus", "center". In the process of studying the dynamic model of a second-order linear system, phase portraits and graphs of transient characteristics were obtained.

1 Introduction

In mathematics and physics, phase space is understood as a space in which each point corresponds to one and only one state from the set of all possible states of the system [1-3]. The point in space corresponding to the state of the system is called "depicting" or "representing" for it. The change in the states of the system, that is, its dynamics, is compared with the movement of the representing point. The trajectory of this point is called the phase trajectory. It is important to note that it is not identical to the actual trajectory of movement. The speed of such a representing point is defined as the phase speed.

The phase space method is one of the main methods for studying the behavior of dynamic systems [4-5]. Applicable for both linear and nonlinear systems, and the qualitative nature of singular points and the behavior of phase trajectories near them are the same for both types of systems [2]. Therefore, the types of singular points and the corresponding phase portraits can be considered using the example of a linear system described by a second-order differential equation:

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$$a_2 y''(t) + a_1 y'(t) + a_0 y(t) = 0 \, , \qquad (1)$$

where a_0, a_1, a_2 – coefficients of a second order differential equation, y – deviation of the output coordinate of the system from the equilibrium state [6].

Let us introduce phase coordinates $y_1 = y_2$ and $y_2 = y_1'$ and rewrite the equation (1) in the form of a system of first order differential equations resolved with respect to derivatives:

$$\frac{dy_1}{dt} = y_2; \frac{dy_2}{dt} = \frac{-a_0}{a_2} y_1 - \frac{a_1}{a_2} y_2 \, . \qquad (2)$$

Obviously, in the plane y_1, y_2 the equilibrium state of the system will correspond to the origin of coordinates. Integrating equations (2) gives the equation of the curves from which phase trajectories are constructed:

$$y_2 = f(y_1, C_1, C_2) \, , \qquad (3)$$

where C_1, C_2 – variable constants.

2 Materials and methods

2.1 Construction of a block diagram of an analog system using operational amplifiers

Let's consider a circuit solution for the formation of phase portraits on functional analog elements. As such, in practice, operational amplifiers have found wide application, which can be used to perform various mathematical operations on analog signals: comparison, algebraic addition, logarithm, integration, differentiation, etc. [7-9]. The resulting structural model provides for the presence of two-phase coordinates – y_1, y_2 (two circuit outputs), what is decisive for the formation of phase portraits and secondary for the formation of transient characteristics. The main structural elements of the circuit are operational amplifiers - analog integrator (1,2), inverting amplifier (3, 4), resistors with variable resistance (5, 6, 7).

Modeling a second-order linear system is possible on an analog device, while the coefficients of the differential equation are adjusted by changing the resistances (5, 6, 7) at the inputs of operational amplifier 1 (Figure 1). Transient processes and phase trajectories are observed on the screen of an electronic oscilloscope (EO). When recording transient processes, it is necessary to turn on the sweep and apply the output coordinate of the model to the vertical input of the oscilloscope. To obtain phase trajectories, the output value of the model y_1 is supplied to the horizontal input of the oscilloscope, and the derivative of y_1, is supplied to the vertical input [6].

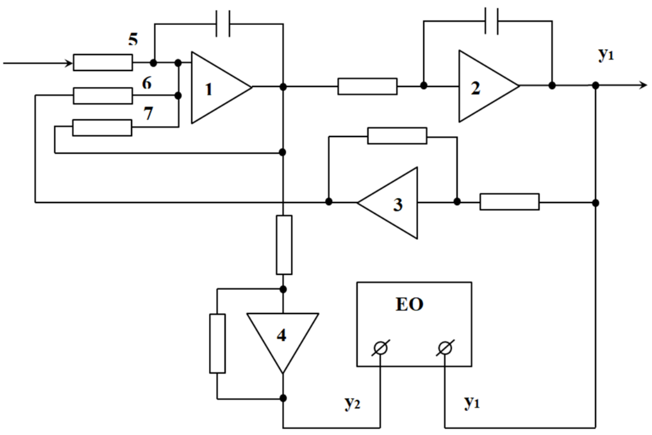


Fig. 1. Block diagram of a second-order linear system based on analog elements.

2.2 Study of a vectorized structural model of an analog system

After vectorizing the structural model of the desired second-order linear system in the Simintech dynamic modeling environment [10, 11] We will study the model using the phase space method, taking into account dynamic characteristics [12]. Let's use elements (dynamic scale Bar) with subsequent binding in the model: input value K, gain A_1, gain A_2. Let us determine the initial conditions for (X_1= -1; X_2=1). Let's set the process completion time to 50 seconds to be able to vary the values of variables (Figure 2).

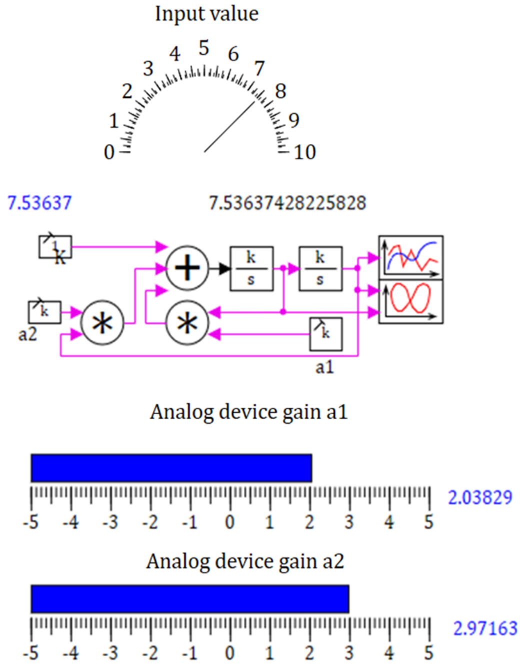


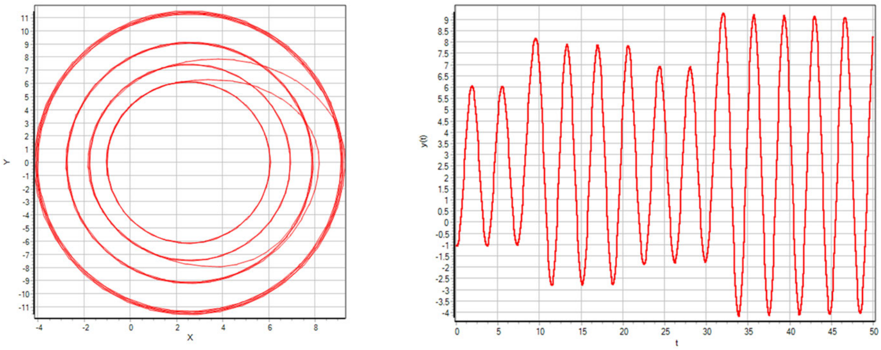
Fig. 2. Interface of a dynamic model of a second order linear system.

As the states under study, we choose the equilibrium state “Center” with the initial coefficients: $a_1 = 0; a_2 = 3$; "Stable Focus" state with initial coefficients: $a_1 = 2; a_2 = 3$; "Unsteady Focus" state with original coefficients $a_1 = -0.5; a_2 = 2$.

3 Results and discussion

3.1 “Center” type equilibrium state

For specific initial odds $a_1 = 0; a_2 = 3$ we get a family of ellipses with a common center. Thus, undamped periodic oscillations correspond to a closed phase trajectory. At the same time, a system in which, under any initial conditions, a periodic oscillation around the equilibrium state occurs (except for the only case when the initial conditions exactly correspond to the equilibrium state) is called a conservative system (a system without energy dissipation). A special point of such a system is called the *center* [6].



a) b)

Fig. 3. Graphic characteristics of the study of the model in the "Center" equilibrium position (a – phase portrait, b – transient process).

The input value K was empirically selected at the level 7.53, which allowed the ellipse to lead to a visual circle. The use of a dynamic data input scale (Bar) allows you to change the model coefficients in real time, which is an imitation of the disturbing influence: with initial data $a_1 = 0; a_2 = 3$ coefficient value a_2 short term changed to value 2 over three iterations.

In this case, the model's response to the phase portrait became visible arcs followed by the drawing of concentric circles (Figure 3).

3.2 Balance state of the "Stable Focus" type

For specific initial odds $a_1 = 2; a_2 = 3$ we get a trajectory that "*flows*" into the origin of coordinates, but the time of movement to the equilibrium state is theoretically infinite. Such a special point is called a stable node. The use of a dynamic data input scale (Bar) allows you to change the model coefficients in real time, which is an imitation of the disturbing influence: with initial data $a_1 = 2; a_2 = 3$ the value of the short-term coefficient a_2 is changed to a value of 0 within six iterations (Figure 4).

At the same time, the response of the model in the phase portrait became loops, the trajectories of which, as they move, flow into a special point - *a stable node*.

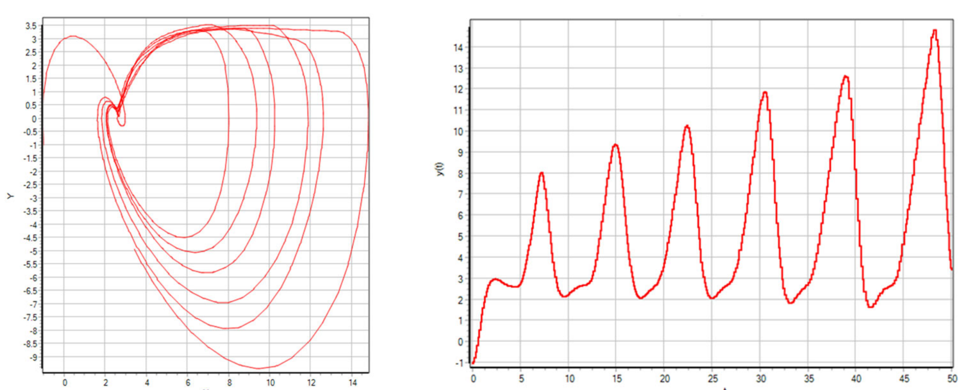


Fig. 4. Graphic characteristics of the study of the model in the equilibrium position "Stable node" (a – phase portrait, b – transient process).

3.3 Equilibrium state of the "Unstable Focus" type

For specific initial odds $a_1 = -0.5; a_2 = 2$ the phase trajectories of such a system look like an unwinding spiral. The representing point, moving along the phase trajectory, moves indefinitely away from the origin of coordinates. The state of equilibrium of the system corresponds to a special point, which is called an *unstable focus*. If the initial conditions correspond to the origin of coordinates, the system will be in a state of equilibrium until it is affected by external disturbances. However, if, as a result of an arbitrarily small disturbance, the system leaves the state of equilibrium, then an oscillatory process with an amplitude will arise. The use of a dynamic data input scale (Bar) allows you to change the model coefficients in real time, which is an imitation of the disturbing influence: with initial data $a_1 = -0.5; a_2 = 2$ coefficient value a_1 short term changed to value 1 over three iterations.

In this case, the model's response to the phase portrait was a short-term focusing of the spiral, followed by unwinding (Figure 5).

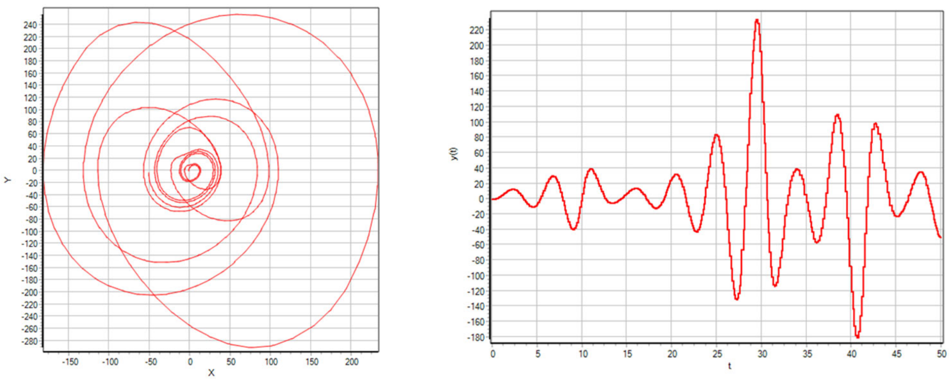


Fig. 5. Graphic characteristics of the study of the model in the equilibrium position "Unstable node" (a – phase portrait, b – transient process).

4 Conclusion

Second-order linear systems are of research interest from the perspective of studying the types of equilibrium positions of the system. Such systems can be represented on analog elements, such as operational amplifiers in the form of a circuit design. The presence of two-phase coordinates allows you to generate both phase portraits and graphs of transient processes. Modern dynamic modeling systems, such as Simintech, allow you to build vectorized models of analog systems. At the same time, dynamic control of gain factors allows one to simulate disturbing influences and study the stability of systems.

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