Hierarchical structures and plane sections of generalized Pascal's pyramid

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Abstract. We research mathematical combinatorial objects of pyramidal structure. Hierarchical structures are constructed based on the sums of the generalized Pascal's pyramid plane sections elements. Enumerative interpretations of these structures constituent objects are obtained, the recurrence relations that these objects satisfy are proved, and some of the most important special cases of the obtained results are considered using the known combinatorial numbers as an example.

1 Introduction

In recent decades, the number of mathematical combinatorial objects and their variety has been steadily growing, which is associated with solving problems of modeling and researching various phenomena and processes. An important direction in the researching of such objects is to consider them as multilevel systems or systems with a hierarchical structure [1-4].

A scheme for constructing combinatorial numbers and polynomials based on a hierarchical pyramidal structure with weights called the generalized Pascal's pyramid is proposed by the authors in the previous researches [5-8].

A number of properties of generalized Pascal's pyramids and their special cases are given in [9]. In [10], based on the generalized Pascal's pyramid, an algorithm for constructing a decision tree was suggested, as well as a method for constructing a search index, which allows to compare different terms based on the weight coefficients of terms and paths.

This work relates to the development of methods for the analysis of hierarchical systems and their applications. We consider a hierarchical system of generalized Pascal's pyramid plane sections elements sums.

The values of sums and enumerative interpretations of the considered hierarchical structures are found. We prove recurrence relations that these sums satisfy, and also give some special cases.

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2 Generalized Pascal's pyramid and triangle

A generalized Pascal pyramid (or *V*-pyramid) is an infinite hierarchical trihedral pyramidal structure with weights whose elements for non-negative integers *n*, *k*, *l* satisfy the recurrent relation

$$V(n,k,l) = \alpha_{n,k-1,l} V(n-1,k-1,l) + \beta_{n,k,l-1} V(n-1,k,l-1) + \gamma_{n,k,l} V(n-1,k,l) \quad (1)$$

with boundary conditions $V(0,0,0) = V_0$, $V(n,k,l) = 0$, if $min(n,k,l,n-k-l) < 0$.
Let's posit the vertex of the *V*-pyramid to the origin of the rectangular Cartesian coordinate system in three-dimensional space, and its elements - to the spatial lattice points of the first octant, which have non-negative coordinates. In this case, we place the numbers *n* along the abscissa axis (*Ox*), *k* - along the ordinate axis (*Oy*), and *l* - along the applicate axis (*Oz*). Thus, it is established a matching between the spatial lattice points and the elements of the *V*-pyramid, which will be bounded by the planes *k* = 0, *l* = 0 and *n* – *k* – *l* = 0.

Figure 1 shows the hierarchical structure of the *V*-pyramid, described by relation (1) in a rectangular Cartesian coordinate system in space.

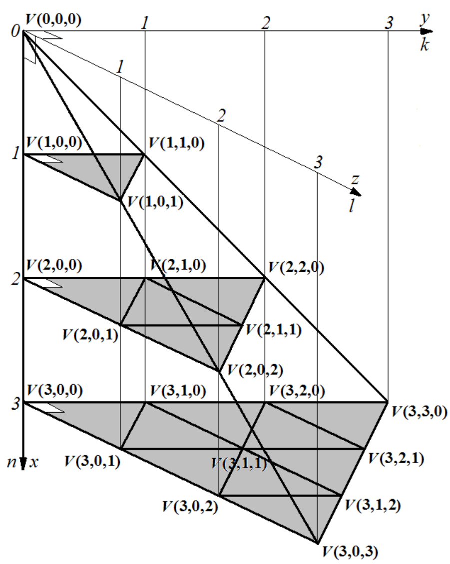

Fig. 1. Hierarchical structure of the V-pyramid.

An important special case of the generalized Pascal's pyramid is the generalized Pascal's triangle (or *V*-triangle), defined as a hierarchical triangular structure with weights, whose elements satisfy the recurrence relation

$$V(n,k) = \alpha_{n,k-1} V(n-1,k-1) + \beta_{n,k} V(n-1,k) \quad (2)$$

with boundary conditions $V(0,0) = V_0$, $V(n,k) = 0$, if $min(n,k,n-k) < 0$.

3 Plane sections of the generalized Pascal pyramid

We consider an arbitrary plane section of the *V*-pyramid, which is some triangle. Let us denote the angles formed by this section with the ordinate and applicate axes as φ and ψ, respectively (Figure 2). That is why the plane section equation is

$$n + tg\,\varphi \cdot k + tg\,\psi \cdot l = const \quad (3)$$

We number all sections of the V-pyramid, which parallel to each other and defined by an equation like (3), starting from the vertex of the pyramid (Figure 3), and consider the sequence $\{S_N(tg\,\phi\,, tg\,\psi)\}$, $N \in \mathbb{N}_0$, of such sections elements sums.

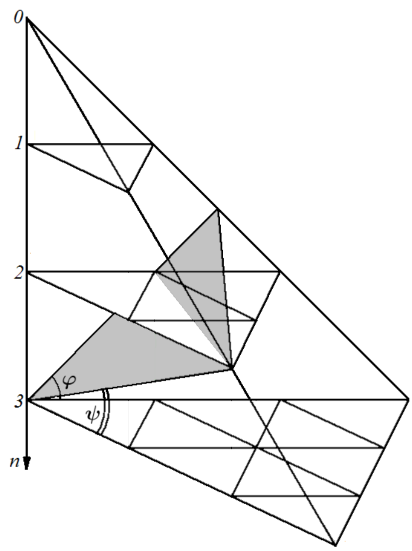

Fig. 2. A plane section of the V-pyramid.

If $tg\,\varphi = \frac{p_1}{q_1}, tg\,\psi = \frac{p_2}{q_2}$, $gcd(p_i, q_i) = 1$, $p_i \in \mathbb{Z}$, $q_i \in \mathbb{N}$, $i = 1{,}2$ and $\frac{p_i}{q_i} > -1$, then the section triangle is finite, and according (3) the equation of the N-th plane section of the V-pyramid is:

$$n + \left(\frac{p_1}{q_1}\right)k + \left(\frac{p_2}{q_2}\right)l = \frac{N}{lcm(q_1, q_2)}. \tag{4}$$

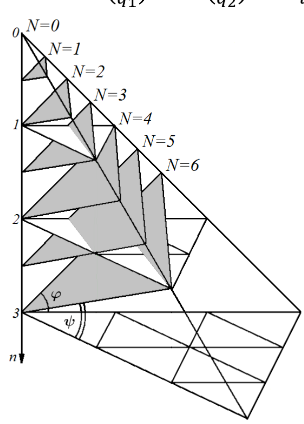

Fig. 3. Parallel to each other plane sections of the V-pyramid.

We denote $q = lcm(q_1, q_2)$ and consider the sum

$$S_N = S_N\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right), p_i \in \mathbb{Z}, q_i \in \mathbb{N}, gcd(p_i, q_i) = 1, \frac{p_i}{q_i} > -1, i = 1{,}2 \tag{5}$$

of elements of the N-th plane section of the V-pyramid.

The sum (5) is calculated by the formula:

$$S_N = \sum_{m=0}^{\left[\frac{q_2 N}{q(p_2+q_2)}\right]} \sum_{r=0}^{\left[\frac{q_1 N}{q(p_1+q_1)}\right]} V\left(\frac{N}{q} - \left(\frac{p_2}{q_2}\right)m - \left(\frac{p_1}{q_1}\right)r, r, m\right). \quad (6)$$

Let X is the sum of elements like $V(n,k,l)$. We introduce the operator $\odot\, \omega_{a,b,c}$, where $a,b,c \in \mathbb{N}_0$, which assigns to each term $V(n,k,l)$ of the sum X the term $\omega_{n+a,k+b,l+c} V(n,k,l)$ of the sum $\odot\, \omega_{a,b,c} X$.

For a partially ordered set $\{a,b,c\}$, we will denote by the symbol $mid(a,b,c)$ its "middle" element, which is not less the minimum and not greater the maximum elements of this set. Let:

$$M_n = min\left(q, \left(\frac{q}{q_1}\right)(p_1+q_1), \left(\frac{q}{q_2}\right)(p_2+q_2)\right),$$

$$M_d = mid\left(q, \left(\frac{q}{q_1}\right)(p_1+q_1), \left(\frac{q}{q_2}\right)(p_2+q_2)\right),$$

$$M_x = max\left(q, \left(\frac{q}{q_1}\right)(p_1+q_1), \left(\frac{q}{q_2}\right)(p_2+q_2)\right).$$

The sequence of sums (5), $N \in \mathbb{N}_0$ satisfies the recurrent relation:

$$S_N = \odot\, \alpha_{1,0,0} S_{N-\left(\frac{q}{q_1}\right)(p_1+q_1)} + \odot\, \beta_{1,0,0} S_{N-\left(\frac{q}{q_2}\right)(p_2+q_2)} + \odot\, \gamma_{1,0,0} S_{N-q} \quad (7)$$

with initial conditions

$$S_0 = V_0, S_1 = S_2 = \ldots = S_{M_n-1} = 0,$$

whereS_I, $I = M_n, \ldots, M_d - 1$, satisfies the recurrent relations

$$S_I = \begin{cases} \odot\, \alpha_{1,0,0} S_{I-\left(\frac{q}{q_1}\right)(p_1+q_1)}, & M_n = \left(\frac{q}{q_1}\right)(p_1+q_1), \\ \odot\, \beta_{1,0,0} S_{I-\left(\frac{q}{q_2}\right)(p_2+q_2)}, & M_n = \left(\frac{q}{q_2}\right)(p_2+q_2), \\ \odot\, \gamma_{1,0,0} S_{I-q}, & M_n = q, \end{cases} \quad (8)$$

where S_J, $J = M_n, \ldots, M_d - 1$, satisfies the recurrent relations

$$S_J = \begin{cases} \odot\, \alpha_{1,0,0} S_{J-\left(\frac{q}{q_1}\right)(p_1+q_1)} + \odot\, \beta_{1,0,0} S_{J-\left(\frac{q}{q_2}\right)(p_2+q_2)}, & M_x = q, \\ \odot\, \alpha_{1,0,0} S_{J-\left(\frac{q}{q_1}\right)(p_1+q_1)} + \odot\, \gamma_{1,0,0} S_{J-q}, & M_x = \left(\frac{q}{q_2}\right)(p_2+q_2), \\ \odot\, \beta_{1,0,0} S_{J-\left(\frac{q}{q_2}\right)(p_2+q_2)} + \odot\, \gamma_{1,0,0} S_{J-q}, & M_x = \left(\frac{q}{q_1}\right)(p_1+q_1). \end{cases} \quad (9)$$

4 Enumerative interpretations of plane sections of the generalized Pascal's pyramid

The sums of elements of the generalized Pascal's pyramid plane sections admit the following interpretation, which generalizes the one given in [10].

A discrete set of variable composition, whose elements are capable of being born, dying, and moving from one qualitative category to another, is called a *population*. Without reducing the generality of the consideration, we believe that initially the population is homogeneous – it consists of identical elements that have two dominant properties: *A* and *B*. We consider the development of the population discretely according to the results of its successive stages, the numbers of which are $N \geq 1$, and according to the double numbers of the elements (r,m) – the degrees of the element possession by the properties *A* and *B* respectively ($r,m \in \mathbb{N}_0$). Let elements of $(r+1,m)$-type be born after t_1 stages of population development, elements of $(r,m+1)$-type be born after t_2 stages of population development, and elements of (r,m)-type be born after t_3 stages of population development ($t_1, t_2, t_3 \in \mathbb{N}$). That is, we are guided by evolution without deviations and with the condition that for t_1, t_2, t_3 stages it is possible to take no more than one step forward. Then the numbers $V\left(\frac{N}{t_3}-\left(\frac{t_2}{t_3}-1\right)m-\left(\frac{t_1}{t_3}-1\right)r,r,m\right)$ and operators $\odot\,\alpha_{1,0,0}$, $\odot\,\beta_{1,0,0}$ and $\odot\,\gamma_{1,0,0}$ in relations (6) – (9) can be interpreted as follows:

$V\left(\frac{N}{t_3}-\left(\frac{t_2}{t_3}-1\right)m-\left(\frac{t_1}{t_3}-1\right)r,r,m\right)$ – the volume (number) of (r,m)-type elements as a result of the N-th stage;

$\odot\,\alpha_{1,0,0}$ – operator producing coefficients $\alpha_{\frac{N}{t_3}-\left(\frac{t_2}{t_3}-1\right)m-\left(\frac{t_1}{t_3}-1\right)r,r,m}$, which are fractions of the volume $V\left(\frac{N}{t_3}-1-\left(\frac{t_2}{t_3}-1\right)m-\left(\frac{t_1}{t_3}-1\right)r,r,m\right)$, in which elements (r,m)-type generated elements of $(r+1,m)$-type after the next t_1 stages of the population development;

$\odot\,\beta_{1,0,0}$ – operator producing coefficients $\beta_{\frac{N}{t_3}-\left(\frac{t_2}{t_3}-1\right)m-\left(\frac{t_1}{t_3}-1\right)r,r,m}$, which are fractions of the volume $V\left(\frac{N}{t_3}-1-\left(\frac{t_2}{t_3}-1\right)m-\left(\frac{t_1}{t_3}-1\right)r,r,m\right)$, in which elements (r,m)-type generated elements of $(r,m+1)$-type after the next t_2 stages of the population development;

$\odot\,\gamma_{1,0,0}$ – operator producing coefficients $\gamma_{\frac{N}{t_3}-\left(\frac{t_2}{t_3}-1\right)m-\left(\frac{t_1}{t_3}-1\right)r,r,m}$, which are fractions of the volume $V\left(\frac{N}{t_3}-1-\left(\frac{t_2}{t_3}-1\right)m-\left(\frac{t_1}{t_3}-1\right)r,r,m\right)$, in which elements (r,m)-type generated exactly the same elements of (r,m)-type after the next t_3 stages of the population development.

The total volume of the population as a result of the N-th stage will be equal to

$$S_N=\sum_{m=0}^{\left[\frac{N}{t_2}\right]}\sum_{r=0}^{\left[\frac{N}{t_1}\right]}V\left(\frac{N}{t_3}-\left(\frac{t_2}{t_3}-1\right)m-\left(\frac{t_1}{t_3}-1\right)r,r,m\right)=$$

$$=\sum_{m=0}^{\left[\frac{(N-t_1)}{t_2}\right]}\sum_{r=1}^{\left[\frac{N}{t_1}\right]}\alpha_{\frac{N}{t_3}-\left(\frac{t_2}{t_3}-1\right)m-\left(\frac{t_1}{t_3}-1\right)r,r-1,m}V\left(\frac{N}{t_3}-1-\left(\frac{t_2}{t_3}-1\right)m-\left(\frac{t_1}{t_3}-1\right)r,r-1,m\right)+$$

$$+\sum_{m=1}^{\left[\frac{N}{t_2}\right]}\sum_{r=0}^{\left[\frac{(N-t_2)}{t_1}\right]}\beta_{\frac{N}{t_3}-\left(\frac{t_2}{t_3}-1\right)m-\left(\frac{t_1}{t_3}-1\right)r,r,m-1}V\left(\frac{N}{t_3}-1-\left(\frac{t_2}{t_3}-1\right)m-\left(\frac{t_1}{t_3}-1\right)r,r,m-1\right)+$$

$$+\sum_{m=0}^{\left[\frac{(N-t_3)}{t_2}\right]}\sum_{r=0}^{\left[\frac{(N-t_3)}{t_1}\right]}\gamma_{\frac{N}{t_3}-\left(\frac{t_2}{t_3}-1\right)m-\left(\frac{t_1}{t_3}-1\right)r,r,m}V\left(\frac{N}{t_3}-1-\left(\frac{t_2}{t_3}-1\right)m-\left(\frac{t_1}{t_3}-1\right)r,r,m\right).\quad(10)$$

If V_0 is given and the weight coefficients of the operators $\odot\,\alpha_{1,0,0}$, $\odot\,\beta_{1,0,0}$ and $\odot\,\gamma_{1,0,0}$ are known, then using formulas (10), (7) - (9) it is possible to calculate the total volume of the population and its inherent distribution by types of elements at each stage of development.

Comment. The inverse problem is not unique if no additional conditions are given.

The proposed interpretation makes it possible to obtain a more compact proof of relations (7) – (9), in contrast to the one given in [10].

Indeed, if $N\geq max(t_1,t_2,t_3)$, then it is possible to born elements of all three types as a result of the N-th stage. In the first case, the population is first formed as a result of the $(N-t_1)$-th stage, and then the element of $(r+1,m)$-type is born. In the second case, the population is first formed as a result of the $(N-t_2)$-th stage, and then the element of $(r,m+1)$-type is born. In the third case, the population is first formed as a result of the $(N-t_3)$-th stage, and then the element of (r,m)-type is born. These possibilities form the recurrent relation (7).

If $mid(t_1,t_2,t_3)\leq N\leq max(t_1,t_2,t_3)-1$, then only two types of elements can be born as a result of the N-th stage. In the first case, the population is first formed as a result of the $(N-mid(t_1,t_2,t_3))$-th stage, and then the element of $(r+1,m)$-type is born, if

$mid(t_1, t_2, t_3) = t_1$ or the element of $(r, m+1)$-type is born, if $mid(t_1, t_2, t_3) = t_2$ or the element of (r, m)-type is born, if $mid(t_1, t_2, t_3) = t_3$. In the second case, the population is first formed as a result of the $(N - min(t_1, t_2, t_3))$-th stage, and then the element of $(r+1, m)$-type is born if $min(t_1, t_2, t_3) = t_1$ or the element of $(r, m+1)$-type is born if $min(t_1, t_2, t_3) = t_2$ or the element of (r, m)-type is born if $min(t_1, t_2, t_3) = t_3$. These possibilities form the recurrent relation (9).

If $min(t_1, t_2, t_3) \leq N \leq mid(t_1, t_2, t_3) - 1$, then it is possible to born elements of only one type as a result of the *N*-th stage. That is, first the population is formed as a result of the $(N - min(t_1, t_2, t_3))$-th stage, and then the element of $(r+1, m)$-type is born if $min(t_1, t_2, t_3) = t_1$ or the element of $(r, m+1)$-type is born if $min(t_1, t_2, t_3) = t_2$ or the element of (r, m)-type is born if $min(t_1, t_2, t_3) = t_3$. These possibilities form the recurrent relation (8).

If $1 \leq N \leq min(t_1, t_2, t_3) - 1$, then it is impossible to born elements of any of the indicated types as a result of the *N*-th stage. Thus, we obtain the initial conditions $S_1 = S_2 = \ldots = S_{M_n - 1} = 0$.

Relations (7) – (9) are proved.

5 Plane sections of the generalized Pascal pyramid and some combinatorial numbers

Let's consider some of the most important special cases.

5.1 Generalized Tribonacci numbers of the first kind

For $tg\,\varphi = -\frac{1}{2}, tg\,\psi = \frac{1}{2}$, $V_0 = 1$, $\alpha_{n,k,l} = \alpha_{n-1}$, $\beta_{n,k,l} = \beta_{n-1}$, $\gamma_{n,k,l} = \gamma_{n-1}$, $n \geq 1$, $0 \leq k + l \leq n$, we have $V(n,k,l) = B^n_{k,l}$ and obtain plane sections of the *B*-pyramid. The equations of the plane of such sections is $n - \left(\frac{1}{2}\right)k + \left(\frac{1}{2}\right)l = N$.

The sums $S_N\left(-\frac{1}{2}, \frac{1}{2}\right)$, $N \in \mathbb{N}_0$, of the elements of these sections, based on (6), is calculated by the formula

$$S_N\left(-\frac{1}{2}, \frac{1}{2}\right) = \sum_{m=0}^{\left[\frac{N}{3}\right]} \sum_{r=0}^{[N]} B_{r,m}^{\frac{(N-m+r)}{2}}$$

and form a sequence that, based on (7) – (9), satisfies the recurrent relation

$$S_N = \oplus_\alpha S_{N-1} + \oplus_\beta S_{N-3} + \oplus_\gamma S_{N-2} \tag{11}$$
$$S_0 = B^0_{0,0}, S_1 = B^1_{1,0}, S_2 = B^1_{0,0} + B^2_{2,0},$$

where $\oplus_\alpha$, $\oplus_\beta$ and $\oplus_\gamma$ are particular cases of the operators $\odot\, \alpha_{1,0,0}$, $\odot\, \beta_{1,0,0}$ and $\odot\, \gamma_{1,0,0}$ for $\alpha_{n,k,l} = \alpha_{n-1}, \beta_{n,k,l} = \beta_{n-1}, \gamma_{n,k,l} = \gamma_{n-1}$.

Note that the recurrent relation (11) defines the generalized Tribonacci numbers of the first kind [10] defined as

$$\tau_1(n) = \sum_{m=0}^{\left[\frac{n}{2}\right]} \sum_{r=0}^{\left[\frac{n}{3}\right]} B^{n-m-2r}_{n-2m-3r,r},$$

and satisfying the following recurrent relation:

$$\tau_1(n) = \otimes_\alpha \tau_1(n-1) + \otimes_\gamma \tau_1(n-2) + \otimes_\beta \tau_1(n-3),$$

with initial conditions $\tau_1(0) = B^0_{0,0}$, $\tau_1(1) = B^1_{1,0}$, $\tau_1(2) = B^2_{2,0} + B^1_{0,0}$.

5.2 Generalized Tribonacci numbers of the second kind

For $tg\,\varphi = -\frac{1}{2}, tg\,\psi = \frac{1}{2}$, $V_0 = 1$, $\alpha_{n,k,l} = \alpha_k$, $\beta_{n,k,l} = \beta_k$, $\gamma_{n,k,l} = \gamma_k$, $n \geq 1$, $0 \leq k + l \leq n$, we have $V(n, k, l) = A^n_{k,l}$ and obtain plane sections of the A-pyramid. The section equation, as in the first case, has the form $n - \left(\frac{1}{2}\right)k + \left(\frac{1}{2}\right)l = N$, but it sets completely different amounts.

The sums $S_N\left(-\frac{1}{2}, \frac{1}{2}\right)$, $N \in \mathbb{N}_0$, of the elements of these sections, based on (6), is calculated by the formula

$$S_N\left(-\frac{1}{2}, \frac{1}{2}\right) = \sum_{m=0}^{\left[\frac{N}{3}\right]} \sum_{r=0}^{[N]} A_{r,m}^{\frac{(N-m+r)}{2}}.$$

and form a sequence that, based on (7) – (9), satisfies the recurrent relation

$$S_N = \oplus_\alpha S_{N-1} + \oplus_\beta S_{N-3} + \oplus_\gamma S_{N-2}, \quad (12)$$
$$S_0 = A^0_{0,0}, S_1 = A^1_{1,0}, S_2 = A^1_{0,0} + A^2_{2,0},$$

where $\oplus_\alpha$, $\oplus_\beta$ and $\oplus_\gamma$ are particular cases of the operators $\odot\, \alpha_{1,0,0}$, $\odot\, \beta_{1,0,0}$ and $\odot\, \gamma_{1,0,0}$ for $\alpha_{n,k,l} = \alpha_k$, $\beta_{n,k,l} = \beta_k$, $\gamma_{n,k,l} = \gamma_k$.

Note that the recurrent relation (12) defines the generalized Tribonacci numbers of the second kind [10] defined as

$$\tau_2(n) = \sum_{m=0}^{\left[\frac{n}{2}\right]} \sum_{r=0}^{\left[\frac{n}{3}\right]} A^{n-m-2r}_{n-2m-3r,r},$$

and satisfying the following recurrent relation:

$$\tau_2(n) = \oplus_\alpha \tau_2(n-1) + \oplus_\gamma \tau_2(n-2) + \oplus_\beta \tau_2(n-3),$$

with initial conditions $\tau_2(0) = A^0_{0,0}$, $\tau_2(1) = A^1_{1,0}$, $\tau_2(2) = A^2_{2,0} + A^1_{0,0}$.

6 Conclusion

This work belongs to the field of combinatorial analysis and is devoted to the study of combinatorial numbers properties and their generalizations by identifying relationships between elements of sections of a spatial combinatorial configuration called the generalized Pascal pyramid. Of course, without a geometric representation of the results, the objects under study, which arose at different times and from tasks of different nature, created the impression of fragmentation. The geometric method of representing combinatorial objects as elements of the generalized Pascal's pyramid sections made it possible to reveal new properties and interpretations of these objects in a unified way.

The obtained results are important for the development of the general theory of combinatorial analysis and for solving applied problems of enumerative combinatorics, and can also be used in the study of complex biological systems. Further study of sections and parts of Pascal's generalized pyramid will make it possible to establish new relationships between known objects, transfer the properties of some objects to others, and also obtain new objects that have not been studied before.

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