Analytical calculation of self-inductance for triangular wires

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Abstract. The paper presents a relatively easy approach for analytical calculation of the self-inductance for wires of a triangular cross-section. The Kalantarov-Zeitlin approach was applied to develop a new set of formulae. The low-frequency and direct current case is studied. The final expressions provide simple and accurate approximation of the self-inductance of the triangular wire and might serve as a basis for further calculation of nontrivial geometries.

1 Introduction

For simplifying inductance calculations the approximation of Geometric Mean Distance (GMD), originated from Maxwell [1], is usually used. The method is usually divided on mutual-GMD for mutual and self-GMD for self-inductance calculations. Self-GMD method is correct for calculating the self-inductance of long straight wires; for curved conductors it only approximates the exact result [2, 3]. The lesser is ratio g/R, where g is self-GMD and R is winding radius, the more exact is the result. It is also known a more exact formula, proposed by Kalantarov and Zeitlin [4], which gives a precision of $(g/R)^2$ order.

Practical formulae cover the most common geometries, i.e. circular or rectangular conductors. More complicated shapes, particularly, a triangular cross section are calculated by numerical methods that imply massive computations.

This paper describes a possibility to calculate the self-inductance of a triangular conductor using relatively simple analytical expressions. The new formulae have been derived using the Kalantarov-Zeitlin approaches [4].

The original formula expresses the self-inductance L of a wire with an arbitrary cross section S as a sum of components related to the wire geometry and current distribution.

$$L = N - G + A - Q, \tag{1}$$

where N depends on the wire length and curvature, while coefficients G, A, and Q involves the cross-sectional geometry and current density distribution.

Equation (1) is accurate to within $(g/2R)^2$ and $(g/l)^2$, where l is the wire length, R is the minimal radius of curvature, g is geometric mean distance of the wire cross section to itself (self-GMD).

The value N is obtained by simple integration over the wire length; it is independent of a cross-section. For this reason, N is assumed a known value.

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G, A, and Q are calculated via integration over the cross section. For the low-frequency case and direct current, they can be written in the SI form as

$$G = \frac{\mu_0}{2\pi} \ln g_{\mathrm{m}} \tag{2}$$

$$A = \frac{\mu_0}{2\pi} a_{\mathrm{m}} \tag{3}$$

$$Q = \frac{\mu_0}{8\pi D} q_{\mathrm{m}}^2 \tag{4}$$

where g_{m}, a_{m} и q_{m} are, respectively, the geometric mean, arithmetic mean, and quadratic mean distances of the wire cross section S from itself. D is the distance between the utmost sides of the wire ($D >> q_{\mathrm{m}}$).

The geometric mean, arithmetic mean, and quadratic mean distances are calculated as

$$\ln g_{\mathrm{m}} = \frac{1}{S^2} \iint_S \ln \eta \, \mathrm{d}s_1 \mathrm{d}s_2 \tag{5}$$

$$a_{\mathrm{m}} = \frac{1}{S^2} \iint_S \eta \, \mathrm{d}s_1 \mathrm{d}s_2 \tag{6}$$

$$q_{\mathrm{m}}^2 = \frac{1}{S^2} \iint_S \eta^2 \mathrm{d}s_1 \mathrm{d}s_2 \tag{7}$$

Here $\mathrm{d}s_1$ and $\mathrm{d}s_2$ are differential of area associated with points P_1 and P_2 taken on the wire cross section S, η is the distance between P_1 and P_2, see Figure 1.

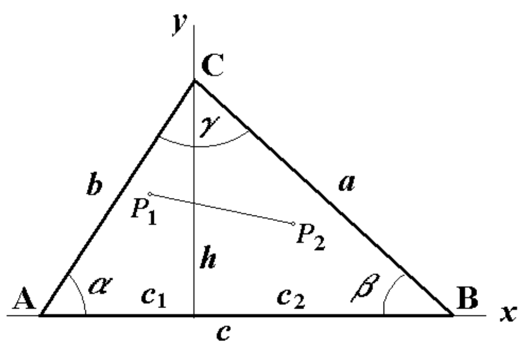

Fig. 1. Base notations: A, B, C – vertices of triangle; a, b, c and α, β, γ – corresponding sides and angles; h and c_1, c_2 define integrals (9); $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$ – some points in (8).

If the cross section lies on the plane xOy, then, using the Cartesian coordinates, we obtain

$$\eta(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{8}$$

where subscripts 1 and 2 indicate respective points.

For ordinary circular or rectangular cross sections, the self-inductance is obtained via double integration (5-7) with respect to coordinates resulting in quadruple one-dimensional integrals with fixed limits [5]. In the case of the triangular cross section, integration limits are variable.

2 Self-inductance of a triangular wire

Figure 1 represents an irregular triangle with sides a, b and c. A Cartesian coordinate system is applied so that the triangle lies in the xOy plane. Vertex C falls on the axis y, side AB is put along the axis x. It may be assumed without losing the generality that vertices A and B are opposite with respect to the origin. The distance from the origin is c_1 for A and c_2 for B.

Then formulae (5-7) can be put into a combined form

$$\{\ln g_\mathrm{m}, a_\mathrm{m}, q_\mathrm{m}^2\} = \frac{1}{S^2}\int_0^h \mathrm{d}y_2 \int_{f_1(y_2)}^{f_2(y_2)} \mathrm{d}x_2 \int_0^h \mathrm{d}y_1 \int_{f_1(y_1)}^{f_2(y_1)} \{\ln\eta, \eta, \eta^2\}\mathrm{d}x_1 \tag{9}$$

where $f_1(y) = c_1(y/h-1)$ and $f_2(y) = c_2(1-y/h)$ are the lower and upper limits of integration with respect to x, the distance η comes from (8).

Quadruple integration of (9) with respect to x_1, y_1, x_2, y_2 gives a set of relatively simple formulae for the triangular cross section.

The geometric mean distance of the triangle from itself is calculated as

$$\ln g_\mathrm{m} = -\frac{25}{12} + \frac{2S}{3}\left(\frac{\alpha}{a^2} + \frac{\beta}{b^2} + \frac{\gamma}{c^2}\right) + \frac{a^2(b^2+c^2)-(b^2-c^2)^2}{6b^2c^2}\ln a$$
$$+ \frac{b^2(a^2+c^2)-(a^2-c^2)^2}{6a^2c^2}\ln b + \frac{c^2(a^2+b^2)-(a^2-b^2)^2}{6a^2b^2}\ln c, \tag{10}$$

where α, β, γ substitutes the expressions (cosine formula)

$$\alpha = \arccos\frac{b^2+c^2-a^2}{2bc},\ \beta = \arccos\frac{a^2+c^2-b^2}{2ac},\ \gamma = \arccos\frac{a^2+b^2-c^2}{2ab},$$

equal to vertex angles (in radians), see figure 1; S substitutes the expression (Heronus formula)

$$S = \frac{1}{4}\sqrt{(b+c-a)(a+c-b)(a+b-c)(a+b+c)},$$

equal to the triangle area S.

The arithmetic mean distance of the triangle from itself is

$$a_\mathrm{m} = \frac{a+b+c}{15} + \frac{(a+b)(a-b)^2}{30c^2} + \frac{(a+c)(a-c)^2}{30b^2} + \frac{(b+c)(b-c)^2}{30a^2}$$
$$+ \frac{4S^2}{15}\left(\frac{1}{a^3}\ln\frac{a+b+c}{b+c-a} + \frac{1}{b^3}\ln\frac{a+b+c}{a+c-b} + \frac{1}{c^3}\ln\frac{a+b+c}{a+b-c}\right) \tag{11}$$

The quadratic mean distance of the triangle from itself is

$$q_\mathrm{m}^2 = \frac{a^2+b^2+c^2}{18} \tag{12}$$

For an equilateral triangle with the side c formulae (10-12) are simplified to the form

$$\ln g_\mathrm{m} = -\frac{25}{12} + \frac{\pi}{2\sqrt{3}} + \ln c \approx -1.176 + \ln c \tag{13}$$

$$a_\mathrm{m} = \frac{4+3\ln 3}{20}c \approx 0.365\cdot c \tag{14}$$

$$q_\mathrm{m}^2 = \frac{c^2}{6} \approx 0.167\cdot c^2 \tag{15}$$

The linear dependence $g_\mathrm{m} = 0.308377\cdot c$, followed from (13), appears to be in an excellent coincidence with the formula $g_\mathrm{m} = 0.308382\cdot c$, numerically evaluated in [6] with using Monte-Carlo method, see Equation (3.2.4) in that report.

Evidently, formulae (10-15) are not sensible to what symbol a, b or c is employed for a triangle side. Hence, the formulae have the symmetrical form with respect to change of side notations. This may be considered as indirect evidence that the obtained formulae are correct. Also, correctness of the analytical formulae (10-15) may be justified numerically.

3 Numerical validation

Let us assume that the vertices of the irregular triangle shown in figure 1 have the following coordinates A = (–1, 0), B = (3, 0), C = (0, 2). Then, $a = \sqrt{13}$, $b = \sqrt{5}$, $c = 4$, $S = 4$, $\alpha = \arccos 5^{-0.5} \approx 1.107$, $\beta \approx 0.588$, $\gamma \approx 1.446$. After substitution of these values to the analytical formulae (10-12), we obtain the results for this triangular wire as listed in the bottom line of table 1.

To verify these results, we may numerically integrate the formulae (9). For this purpose every side of the triangle is divided in two halves. The midpoints are then connected by lines forming 4 identical triangular subareas. Each subarea, in turn, is divided into small triangles in the same way, as shown in Figure 2. The iteration procedure is repeated arriving, generally, to infinitesimal triangular subareas [6].

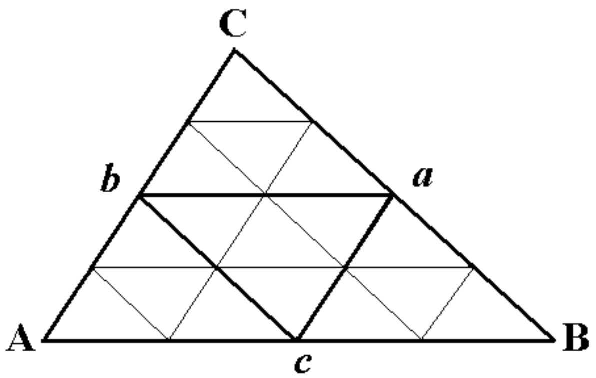

Fig. 2. Iteration scheme. Thick lines for the first subdivision $K = 4$, thin lines for the second subdivision to $K = 16$.

At each step the barycenter is calculated for every sub-area:

$$x_k = \frac{x_{Ak} + x_{Bk} + x_{Ck}}{3}\,,\;\; y_k = \frac{y_{Ak} + y_{Bk} + y_{Ck}}{3}\,,$$

where $k = 1\ldots K$ is the number of sub-triangle, *Ak*, *Bk*, and *Ck* point out its vertices.

After that, partials sums are iteratively calculated

$$\{\ln g_{\mathrm{m}}, a_{\mathrm{m}}, q_{\mathrm{m}}^2\} = \Delta S^2 \sum_{k=1}^{K} \sum_{i \neq k}^{K} \{\ln \eta_{ik}, \eta_{ik}, \eta_{ik}^2\} \tag{16}$$

where $\Delta S = S/K$, η_{ik} is the distance (8) between centers of *i*-th and *k*-th sub-triangles.

Table 1 presents the results (16) of numerical calculations with increasing of *K*. The derived formulae enable quite simple assessment of the self-inductance of a triangular and may significantly reduce computation efforts required with numerical methods.

Table 1. Analytical results obtained with formulae (10-15) vs. numerical integrals (16).

K	$\ln g_{\mathrm{m}}$	a_{m}	q_{m}^2
4	+0.148179	0.972131	1.416667
4^2	+0.069880	1.152456	1.770833
4^3	+0.019471	1.189046	1.859375
4^4	–0.001078	1.197132	1.881510
4^5	–0.008227	1.199024	1.887044
4^6	–0.010520	1.199482	1.888428
4^7	–0.011220	1.199594	1.888774
4^8	–0.011426	1.199622	1.888860
4^9	–0.011486	1.199629	1.888882
formulae (10-15)	**–0.011510**	**1.199631**	**1.888889**

4 Conclusion

Though not much common, conductors with a triangular cross section find use in some practical applications. The proposed approach may also be adapted for other nontrivial geometries in case of direct current. For instance, TF coils of the ITER tokamak utilize conductors with cross sections shaped as a right-angular trapezoid. Applying triangulation, we obtain a combination of a rectangular and two identical triangles. The coil centerline is formed with a straight portion and six circular arcs. Their mutual inductances can be calculated with well-proven formulae [7].

References

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