Conformally coupled massless scalar field in semi-classical gravity and its cosmological consequences.

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Abstract. Recent observations predict accelerated expansion of the universe. To explain this scenario of the universe, different cosmological models have been developed. Initially, General Relativity (GR) theory which is governed by the Einstein field equations was considered to counter this issue. But later on some unexplained issues led the researchers to explore other avenues and as a result they have tried to modify the GR. Einstein's field equation comprises of mainly two parts namely matter part and gravity part. Some of the scientists have modified the matter part, others have modified the gravity part and some of them have modified both. Semi-classical gravity essentially approximates quantum gravity. In semi classical gravity, gravity is assumed to be classical and matter is described using quantum mechanics. In this framework, matter is represented by quantum field in curved spacetime and gravity is modeled by spacetime metric that follows field equations, developed by Einstein. In semi classical gravity, the matter part in the field equations is typically the probabilistic expected value of the energy-momentum tensor operator.
In this work, we have contemplated a homogeneous and isotropic FRW model of the universe with dark matter and dark energy. Choosing the dark energy in the form of mass less conformally coupled scalar field in semi classical gravity, we have got the evolution equations which have been transformed into an autonomous system by considering suitable dimensionless variables. We have tried to look into the stability criterion of universe around the critical points and finally cosmological implications of the behaviour of the critical points have been discussed.

1 Introduction

Current observational data [1,2] strongly suggests an accelerated expansion [3,4] of the universe. Initially this was explained using the cosmological constant Λ [5,6] where the pressure p and energy density ρ are related by $p=-\rho$ which also aligns with the current observational value [7] of the equation of state parameter ω. ω is defined through the

equation $\omega = \frac{p}{\rho}$. However, the cosmological constant fails to address two significant problems, "fine tuning problem" and "Coincidence Problem.
To resolve this issues, scientists introduced a new concept, called "dark energy"[8,9] within the frame work of General Relativity. Dark energy is presumed to possess positive energy density and negative pressure, with vacuum energy which is often considered as the source of dark energy [10,11]. To address the problems pertaining to cosmological constant Λ, we typically assume that vacuum energy is balanced by an unknown cancellation mechanism and there is dark energy component associated with variable ω. Different values of this parameter leads to different possible future fates of the universe. For instance, if $\omega > -1$, dark energy represents quintessence fluid. For $\omega = -1$, dark energy corresponds to the cosmological constant and the universe follows a asymptotically de-sitter nature. If $\omega < -1$, General relativity suggests the existence of "Phantom energy". As the university expands ($\omega < -\frac{1}{3}$), the "phantom energy" increases corresponding to proper time and hence the phantom energy density [12] becomes infinite causing the universe to expands infinitely. This would lead to the eventual disintegration of all bound objects like gigantic clusters of galaxies as well as minuscule atomic nuclei and as a result the universe approaches to a future singularity [12] which is classified as the "big rip". To resolve different sorts of singularity and different issues faced in very early cosmological models like big bang theory, inflation theory, cosmological constant; different theoretical models of dark energy have been developed like Scalar field models including Unified dark energy and matter, k-essence, Quintessence, Coupled dark energy and matter etc [13,14].
Several physicists have thought of combining the quantum theory of gravity to eradicate future singularity [15] because they experienced that the quantum theory and general relativity are quite incompatible. As a result they proposed a semi classical theory where they made amalgamation between classical theories of gravity with quantum field theory [16,17]. In classical gravity, the Einstein field equation governs the dynamics of space-time

$$G^{\mu\upsilon} = \frac{8\pi G}{c^4} T^{\mu\upsilon} \quad (1)$$

where left side of the equation (1) represents the gravity part which influences the space time geometry and right hand part represents energy-momentum tensor, analogous to the matter part of universe.
Within quantum gravity theory [18], gravity is represented classically but the matter part is represented quantum mechanically which leads to semi classical Einstein field equation

$$G^{\mu\upsilon} = \frac{8\pi G}{c^4} \left\langle \hat{T}^{\mu\upsilon} \right\rangle \quad (2)$$

where energy-momentum tensor $T^{\mu\upsilon}$ is substituted by quantum operator $\hat{T}^{\mu\upsilon}$ whose expected value represents mass and energy.
Energy momentum tensor plays a crucial role for becoming the origin of curvature in General Relativity. Energy momentum tensor is divergenceless, signifying the conservation of energy and momentum. This symmetry is preserved when transitioning from classical to quantum theory. In massless theories, there is an interesting symmetry known as conformal symmetry, which is reflected by the tracelessness of energy-momentum tensor in classical case. However, in quantum scenario, this is not always true and can be identified through trace anomalies [19,20].

In this study, we focus on the massless scalar field which is conformally coupled [21-22], characterized by an enhanced stress-energy tensor, which can classically contravene null energy condition [21]. In Section 2, we introduce the semi-classical dynamical equations for this massless conformally coupled scalar field. Section 3 discusses the dynamical system analysis and its application towards qualitative behavior of cosmological models. Section 4 elaborates on the incorporation of this dynamical system tool in our current model and presents a detailed analysis.

2 Semi classical dynamical equations in conformally coupled massless scalar field

Here, let us explore the modified field equations in conformally coupled massless scalar field. We are considering here a homogeneous and isotropic FRW metric

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\, d\Omega^2\right] \tag{3}$$

where $d\Omega^2 = \left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$

As it is known that in semi classical gravity for a mass less scalar field which is conformally coupled, the energy momentum tensor is not traceless [22].Vacuum trace anomaly of energy momentum tensor can be denoted as

$$T_{vac} = \frac{1}{2880\pi^2}(\nabla R + \frac{1}{2}G) \tag{4}$$

Here, R is scalar curvature and $G = -2\left(R_{\mu\upsilon}R^{\mu\upsilon} - \frac{1}{3}R^2\right)$ is Gauss-Bonnet curvature invariant. Here, considering that Weyl tensor diminishes in Friedmaan-Robertson-Walker geometry for deriving G.

By substituting Hubble parameter $H = \frac{\dot{a}}{a}$ in (4) where $a(t)$ is the scale factor, one can deduce

$$T_{vac} = \frac{1}{480\pi^2}\left(\dddot{H} + 12H^2\dot{H} + 7H\ddot{H} + 4\dot{H}^2\right) + \frac{1}{240\pi^2}\left(H^4 + H^2\dot{H}\right) \tag{5}$$

Here, we use the trace anomaly $T_{vac} = \rho_{vac} - 3p_{vac}$ (6)

After inserting (5) in $\dot{\rho}_{vac} + 3H\left(\rho_{vac} + p_{vac}\right) = 0$ (6), known as the conservation equation [12], vacuum energy density becomes

$$\rho_{vac} = \frac{1}{480\pi^2}\left(3H^2\dot{H} + H\ddot{H} - \frac{1}{2}\dot{H}^2\right) + \frac{1}{960\pi^2}H^4 \tag{7}$$

The semi classical Friedmann equation can be defined here as

$$H^2 = \frac{8\pi}{3}\left(\rho + \rho_{vac}\right) \tag{8}$$

3 Dynamical system analysis

The term "Dynamical system" is used in an informal sense to denote a physical system which evolves with respect to time. It is used in a precise mathematical sense in two ways: continuous and discrete dynamical system.
It is assumed that the state of a physical system at any instance of time can be represented by an element u of a state space U which may be either finite or infinite dimensional.Following set of autonomous differential equation represents the evolution of the system. [23]

$$\frac{du}{dt} = f(u),\ u \in U, \text{ where } f : U \to U \tag{9}$$

If U is finite dimensional the (9) represents an autonomous system of ordinary differential equation and if U is infinite dimensional then (10) describes an autonomous system of partial differential equation.
For autonomous system of ODE $\dot{U} = f(U),\ U \epsilon R^n$ and for a given flow ϕ_t, the points in the state space can be classified into two different types, singular or equilibrium points and ordinary points. A singular point of an ODE is considered hyperbolic if the real part $\mathrm{Re}(\lambda_i) = 0$ for all eigen values λ_i corresponding to the linearized Jacobian matrix pertaining to the vector field $f(u)$ determined [23,24] at singular points; otherwise, it is defined as non-hyperbolic. Near the local neighbourhood of a singular point, behaviour of the system is analysed by linear approximation of the vector field, based on Hartman-Grobman theorem. Mathematically, $f(u) \approx Df(\bar{u})(u - \bar{u})$, $Df(\bar{u})$ represents the linearized Jacobian matrix pertaining to the vector field at the singular point $\bar{u}$ [23,24]. For a linear system of ODEs, the singular point may be Source, Sink or Saddle point. The phase space of a linear system of ODEs is spanned by the eigenvectors of the linearized matrix and the eigen spaces divide the phase space into three subspaces: the stable subspace, unstable subspace, and central subspace.

3.1 Dynamical System in cosmology

Field equations corresponding to Einstein's General relativity represent a highly complex set of non-linear differential equations that describes universe's evolution. Solving these equations analytically is often difficult or sometimes impossible due to the non-linear nature of the system. In such cases, tools from dynamical system analysis become invaluable for gaining insights into the system's behaviour. The cosmological equations of motion are reduced to a self-autonomous system, $\dot{U} = f(U)$ where U represents the auxiliary variables and $f(U)$ defines the autonomous system. Here "." denotes the differentiation corresponding to log a(t). Here a(t) refers to scale factor.

Critical points U_c are obtained from the system $f(U)=0$ and by applying perturbation techniques, a matrix equation $X = VX$ is derived. In this equation, X represents perturbations of variables and coefficients of the perturbation equations are denoted through the matrix V. The eigenvalues are derived from the matrix V corresponding to the critical points U_c. Nature and stability around those critical points are determined by certain conditions related to the eigenvalues and determinant of V i.e. $tr(V) < 0$ and $\det(V) > 0$.

Cosmological models are then studied qualitatively with viable solutions checked against different constraints to analyse the system's behaviour. A feasible cosmological solution suggests that our current universe is a global attractor, meaning that all possible initial conditions converge to the observed probable ratio of dark energy, dark matter. As a result the stable critical points, which serve as attractors, is rigorously analysed.

4 Dynamical system analysis in semi classical gravity for a conformally coupled massless scalar field.

In this current cosmological model, we need to frame a system of autonomous equations using semi classical Friedmann equations and conservation equation (5), (6), (7), (8), (9) to use dynamical system tools. To construct such self-autonomous system, we consider the following dimensionless variables

$$\overline{H} = \frac{H}{H_+}, \overline{Y} = \frac{\dot{H}}{{H_+}^2} \quad \text{and} \quad \overline{\rho} = \frac{8\pi\rho}{3{H_+}^2} \quad \text{where } H_+ = \sqrt{360\pi} \tag{10}$$

Using (11) in (7), (8) and (9), we obtain following system of differential equations:

$$\overline{H}' = \overline{Y} \tag{11}$$

$$\overline{Y}' = \frac{1}{2\overline{\overline{H}}}\left(\overline{H}^2 - \overline{\rho} - 6\overline{H}^2\overline{Y} + \overline{Y}^2 - \overline{H}^4\right) \tag{12}$$

$$\overline{\rho}' = -3\overline{H}(1+\omega)\overline{\rho} \tag{13}$$

where derivative corresponding to $\bar{t}$ is denoted by "ʹ ".

To solve the above autonomous system we frame a linearized matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 - 6\overline{Y} - 2\overline{H}^2 - \frac{1}{2\overline{H}^2}\left(\overline{H}^2 - \overline{\rho} - 6\overline{H}^2\overline{Y} + \overline{Y}^2 - \overline{H}^4\right) & -3\overline{H} + \frac{\overline{Y}}{\overline{\overline{H}}} & -\frac{1}{2\overline{\overline{H}}} \\ -3(1+\omega)\overline{\rho} & 0 & -3\overline{H}(1+\omega) \end{pmatrix} \tag{14}$$

Corresponding to all critical points obtained from (11), (12) and (13), eigen values are fetched from the linearized Jacobian matrix (14) which are given in Table-1

Table 1. Eigen values corresponding to all critical points obtained from the autonomous system.

Critical Point $\left(\overline{H},\overline{Y},\overline{\rho}\right)$	**Eigen Values**
(1,0,0)	$-\left(3+3\omega\right),\ \frac{-3-\sqrt{5}}{2},\ \frac{-3+\sqrt{5}}{2}$
(−1,0,0)	$\left(3+3\omega\right),\frac{-3-\sqrt{5}}{2},\frac{-3+\sqrt{5}}{2}$

To identify the qualitative nature of the system around the critical points we explore a two dimensional phase plot diagram in $\left(\overline{H},\ \overline{Y}\right)$ and $\left(\overline{H},\overline{\rho}\right)$ plane.

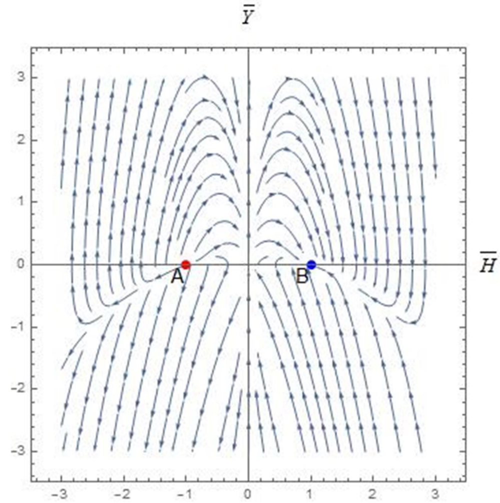

Fig. 1. Phase plot corresponding to the critical point $A(-1,0)$ and $B(1,0)$ in $\left(\overline{H},\overline{Y}\right)$ plane.

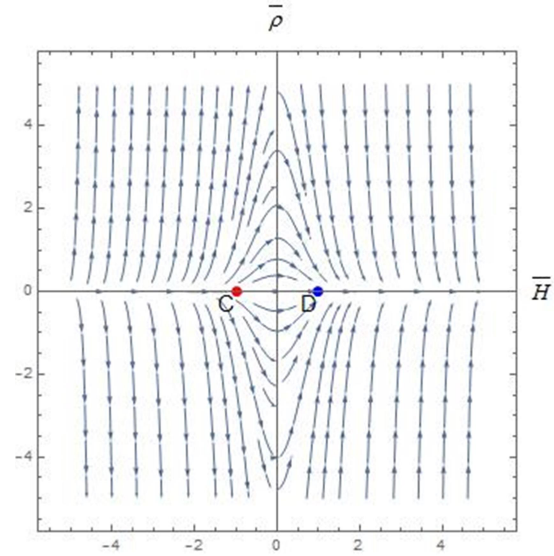

Fig. 2. Phase plot corresponding to the point $C(-1,0)$ and $D(1,0)$ for $\omega = -0.85$ in $(\overline{H}, \overline{\rho})$ plane.

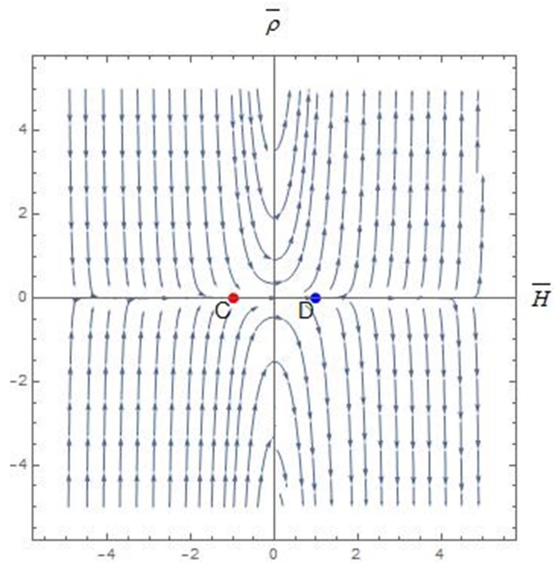

Fig.3. Phase plot corresponding to the point $C(-1,0)$ and $D(1,0)$ for $\omega = -1.25$ in $(\overline{H}, \overline{\rho})$ plane.

5 Conclusion

In this work, we have not considered any particular fluid description like baryon, radiation and dust, rather we have assumed general description of fluid. Here, pertaining to semi classical gravity with massless conformally coupled scalar field, energy density is completely specified in the terms of trace anomaly. From Table-I, we can easily note that (1,0,0) will be a stable non hyperbolic node if the equation of state parameter lies within $-1 < \omega < -\frac{1}{3}$, which indicates that the energy density considered here represents quintessence description of fluid with an accelerated phase. Phantom era is ruled out.
(-1,0,0) always represents a saddle node i.e. unstable irrespective of any particular choice of ω. Thus we discard this point for representing universe.
We can also visualize the aforesaid fact from the phase plot diagram. In Fig.1, we have shown a two dimensional $\left(\overline{H}, \overline{Y}\right)$ plane where we have considered parametric value of vacuum energy density which depicts that A is becoming a stable node and B is a saddle one.
In Fig.2 and Fig.3, we have considered a two dimensional phase plane in $\left(\overline{H}, \overline{\rho}\right)$ and we have taken two values of equation of state parameter ω as -0.85 and -1.25 respectively. The outcome is also inclined to our analysis in terms of the eigen values, obtained with respect to both the critical points. From Fig.2 we can easily see that whenever ω is taken in the accelerated quintessence range, D is becoming stable and C is becoming saddle whereas when ω has been chosen from the phantom region, D is becoming unstable.
Thus from our current work, we can vouch that semi classical gravity with conformally coupled scalar field rules out the present universe to be in phantom era, rather we are in a quintessence era with an accelerating phase.

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