Zernike by ONE *Pascal* triangle

A high performance, low memory cost and flexible computation scheme for *Zernike* polynomials

Wei–Jun Chen[1, *]

[1]Carl Zeiss Meditec AG

Abstract. This work discovers two hidden cases of blockwise recurrence in *Zernike* computations. Based on these findings, a new computation scheme for *Zernike* polynomials is proposed. It uses one *Pascal* triangle for all internal factors, thus avoiding calculations of factorials, cos/sin, inverse matrix, etc., and meets the requirements of computational accuracy, high speed, low memory footprint, and flexibility.

1 Introduction

Zernike polynomials play a key role in many optical applications[1] due to their two main features: 1. the orthogonality between individual polynomial components and 2. the direct correspondence with optical *Seidel* aberrations. If the polynomial order is not high, the calculation of *Zernike* polynomials is usually satisfied by the following standard formula:

$$\begin{aligned} z_n^{\pm m}(\rho,\varphi) &= R_n^m(\rho)\begin{cases} \cos(m\varphi) & \text{for} \quad +m \\ \sin(m\varphi) & \text{for} \quad -m \end{cases}, \\ R_n^m &= \sum_{k=0}^{(n-m)/2}\left(\frac{(-1)^k(n-k)!}{k!\left(\frac{n+m}{2}-k\right)!\left(\frac{n-m}{2}-k\right)!}\rho^{n-2k}\right), \end{aligned} \tag{1}$$

where R_n^m is called the *Zernike* radial polynomial, for which a recurrence relation has been discovered and applied to recursive *Zernike* computations of higher order [2] [3]:

$$R_n^m = \rho\Big(R_{n-1}^{|m-1|}(\rho) + R_{n-1}^{m+1}(\rho)\Big) - R_{n-2}^m(\rho). \tag{2}$$

On the other hand, *Zernike* polynomials are essentially one of the orthogonalized versions of the homogeneous bivariate polynomials in a *Cartesian* coordinate system with a unit circle and a coordinate center. In practice, the computation of *Zernike* polynomials is often based on a look-up-table of the corresponding *Cartesian* forms of the individual *Zernike* components[1] [4] [5], although the complexity of such *Cartesian* forms increases sharply with polynomial order.

With the aim of achieving fewer accumulation errors and more flexibility than in recursive computations, while lowering the limits of polynomial order and complexity in *Cartesian* form based computations, this paper investigates in the following section a new calculation scheme, starting from the less discussed binomial form of the *Zernike* definition (Eq.3).

*e-mail: wei-jun.chen@zeiss.com

2 Zernike by one Pascal

Zernike radial polynomial has its binomial form

$$R_n^m = \sum_{k=0}^{(n-m)/2} (-1)^k \binom{n-k}{k} \binom{n-2k}{\frac{n-m}{2}-k} \rho^{n-2k}. \tag{3}$$

Defining $n = 2j, m = 2l$ for even orders and $n = 2j+1, m = 2l+1$ for odd orders, we get

$$\begin{cases} R_{2j}^{2l} = \sum_{k=0}^{j-l} (-1)^k \binom{J_{2k}+k}{J_{2k}} \binom{J_{2k}}{J_k+l} (\rho^2)^{J_k} \\ R_{2l+1}^{2l+1} = \rho \sum_{k=0}^{j-l} (-1)^k \left(\binom{J_{2k}+k}{J_{2k}} + \binom{J_{2k}+k}{J_{2k}+1} \right) \left(\binom{J_{2k}}{J_k+l} + \binom{J_{2k}}{J_k+l+1} \right) (\rho^2)^{J_k} \end{cases} \tag{4}$$

where $J_k = j-k$ and $J_{2k} = 2J_k$. If we further define $J_{kl} = j-k-l$ and combine *de Moivre*'s formula $(\cos m\varphi + i \sin m\varphi) = (\cos\varphi + i \sin\varphi)^m$, we get

$$\begin{cases} z_{2j}^{2l} = \sum_{k=0}^{j-l} (-1)^k w_k^e t_{lk}^e \left(\left(\sum_{q=0}^{J_{kl}} b_{J_{kl}}^q (x^2)^q (y^2)^{J_{kl}-q} \right) \times \left(\sum_{p=0}^{l} (-1)^{l-p} b_{2l}^{2p} (x^2)^p (y^2)^{l-p} \right) \right) \\ z_{2j}^{-2l} = -xy^{-1} \sum_{k=0}^{j-l} (-1)^k w_k^e t_{lk}^e \left(\left(\sum_{q=0}^{J_{kl}} b_{J_{kl}}^q (x^2)^q (y^2)^{J_{kl}-q} \right) \times \left(\sum_{p=0}^{l-1} (-1)^{l-p} b_{2l}^{2p+1} (x^2)^p (y^2)^{l-p} \right) \right) \\ z_{2j+1}^{2l+1} = x \sum_{k=0}^{j-l} (-1)^k w_k^o t_{lk}^o \left(\left(\sum_{q=0}^{J_{kl}} b_{J_{kl}}^q (x^2)^q (y^2)^{J_{kl}-q} \right) \times \left(\sum_{p=0}^{l} (-1)^{l-p} b_{2l+1}^{2p+1} (x^2)^p (y^2)^{l-p} \right) \right) \\ z_{2j+1}^{-2l-1} = y \sum_{k=0}^{j-l} (-1)^k w_k^o t_{lk}^o \left(\left(\sum_{q=0}^{J_{kl}} b_{J_{kl}}^q (x^2)^q (y^2)^{J_{kl}-q} \right) \times \left(\sum_{p=0}^{l} (-1)^{l-p} b_{2l+1}^{2p} (x^2)^p (y^2)^{l-p} \right) \right) \end{cases}, \tag{5}$$

where $b_a^b = \binom{a}{b}$ denotes a binomial factor, and we have $w_k^e = \binom{J_{2k}+k}{J_{2k}}$, $w_k^o = w_k^e + \binom{J_{2k}+k}{J_{2k}+1}$, $t_{lk}^e = \binom{J_{2k}}{J_k+l}$, and $t_{lk}^o = t_{lk}^e + \binom{J_{2k}}{J_k+l+1}$, respectively. All these factors can be retrieved from one n-row *Pascal* triangle.

Eq. 5 shows the recurrence of binomial factors and the recurrence of multiplication of two polynomials, based on which a new computation scheme for *Zernike* is proposed, which offers better accuracy, speed, memory consumption and flexibility.

References

[1] Kuo Niu and Chao Tian, Zernike polynomials and their applications, *Journal of Optics*, **vol. 24(123001)**, 1–54 (2022)

[2] B. H. Shakibaei and R. Paramesran, Recursive formula to compute Zernike radial polynomials, *Optical Letters*, **vol. 38(14)**, 2487–2489 (2013)

[3] Torben B. Andersen, Efficient and robust recurrence relations for the Zernike circle polynomials and their derivatives in Cartesian coordinates, *Optics Express*, **vol. 26(15)**, 18878–18896 (2018)

[4] R. J. Mathar, Zernike basis to Cartesian Transformations, *Serbian Astronomical Journal*, **vol. 179**, 107–120 (2009)

[5] Hedser van Brug, Efficient Cartesian representation of Zernike polynomials in computer memory, *Proceedings SPIE: Fifth International Topical Meeting on Education and Training in Optics* (1997) 382–392