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# Evolution of structure functions at NLO

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**Abstract.** We develop the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) evolution of the Deep Inelastic Scattering (DIS) structure functions at next-to-leading order (NLO) in $\alpha_s$, formulated directly in terms of the structure functions rather than parton distribution functions (PDFs). We refer to this framework as the physical-basis approach. First, we express the NLO PDFs in terms of the structure functions in momentum space. Substituting these expressions into the DGLAP equations, we derive the evolution equations in the physical basis. We demonstrate that, within this approach, the evolution equations are independent of the choice of factorization scale and scheme. Finally, we discuss the application of the NLO physical basis in calculation of LHC cross sections.

## 1 Introduction

Parton distribution functions (PDFs) have long been a cornerstone in the theoretical description of collider physics. However, they are not physical observables and depend on the arbitrary choice of factorization scheme and scale. As the upcoming Electron-Ion Collider (EIC) [1] will produce new experimental data for Deep Inelastic Scattering (DIS) cross sections, reducing theoretical uncertainties in the predictions of the DIS structure functions has become increasingly important.

One approach to decrease the theoretical uncertainty, is to formulate the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) evolution directly for the structure functions, instead of PDFs. We refer to this approach as the physical basis. The concept was first introduced about forty years ago in Ref. [2], and it has been discussed more recently for example in Refs. [3–9]. Previous studies, however, have been limited either to Mellin space or to only one or two observables. In our approach, we formulate the DGLAP evolution in the physical basis directly in momentum space – first at leading order (LO) in $\alpha_s$ [10], and then at next-to-leading (NLO) order [11]. In both cases, the physical basis consist of six observables.

## 2 Establishing a physical basis at NLO

### 2.1 Example with two observables

As a starting point, it is useful to consider a simplified version of the physical basis involving only two observables. In this "singlet approach", we have two structure functions $F_2$ and $F_L$

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which are defined by

$$F_{2,\mathrm{L}}(x,Q^2) = \sum_{j=g,\Sigma} C_{F_{2,\mathrm{L}}f_j}(Q^2,\mu_f^2,\mu_r^2) \otimes f_j(\mu_f^2)\,, \tag{1}$$

where the PDFs are the gluon PDF $g(x,\mu_f^2)$ and the quark singlet over light flavours:

$$\Sigma(x,\mu_f^2) = \sum_q^{n_\mathrm{f}} \left[q(x,\mu_f^2) + \overline{q}(x,\mu_f^2)\right]. \tag{2}$$

Here $n_\mathrm{f} = 3$ is the number of the active quark flavours. To obtain the physical-basis evolution, we are first required to invert the linear mapping in Eq. (1). However, this is a non-trivial task due to the convolutions. We are able to do the inversion perturbatively at NLO – which will be discussed more in the next section – and obtain expressions:

$$f_j(\mu_f^2) = \sum_{i=2,\mathrm{L}} C^{-1}_{F_i f_j}(Q^2,\mu_f^2,\mu_r^2) \otimes F_i(Q^2) + \mathcal{O}(\alpha_\mathrm{s}^2)\,, \tag{3}$$

which we call the physical-basis counterparts for the PDFs. By taking the $\log\left(Q^2\right)$ derivative of Eq. (1)

$$\frac{\mathrm{d}F_i(x,Q^2)}{\mathrm{d}\log(Q^2)} = \sum_{j=g,\Sigma} \frac{\mathrm{d}C_{F_i f_j}(Q^2,\mu_f^2,\mu_r^2)}{\mathrm{d}\log(Q^2)} \otimes f_j(\mu_f^2) \tag{4}$$

and substituting here the physical-basis counterparts of PDFs from Eq. (3), we arrive at a set of DGLAP evolution equations for the structure functions,

$$\begin{aligned} \frac{\mathrm{d}F_i(x,Q^2)}{\mathrm{d}\log(Q^2)} &= \sum_j \frac{\mathrm{d}C_{F_i f_j}(Q^2,\mu_f^2,\mu_r^2)}{\mathrm{d}\log(Q^2)} \otimes \sum_k C^{-1}_{F_k f_j}(Q^2,\mu_f^2) \otimes F_k(Q^2) + \mathcal{O}(\alpha_\mathrm{s}^3) \\ &\equiv \sum_k \mathcal{P}_{ik}(\mu_r^2) \otimes F_k(Q^2) + \mathcal{O}(\alpha_\mathrm{s}^3)\,, \end{aligned} \tag{5}$$

where $\mathcal{P}_{ik}$ are the evolution kernels. Here the renormalization scale $\mu_r$ in the strong coupling $\alpha_\mathrm{s}$ is the only remaining unphysical scale, as the dependence on the factorization scale and scheme cancel within the evolution kernels.

### 2.2 Inverting PDFs at NLO

In order to demonstrate the perturbative inversion which leads to the physical-basis counterparts for PDFs, let us discuss a case without quarks. In this approach, the longitudinal structure function at NLO is defined as

$$\widetilde{F}_\mathrm{L} = C^{(1)}_{F_\mathrm{L}g} \otimes g + \frac{\alpha_\mathrm{s}}{2\pi} C^{(2)}_{F_\mathrm{L}g} \otimes g\,, \tag{6}$$

where

$$\widetilde{F}_\mathrm{L}(x,Q^2) \equiv \frac{2\pi}{\alpha_\mathrm{s}} \frac{F_\mathrm{L}(x,Q^2)}{x}\,. \tag{7}$$

The inversion of the LO coefficient function $C^{(1)}_{F_\mathrm{L}g}$ is

$$g(x) = \hat{P}(x)\left[C^{(1)}_{F_\mathrm{L}g} \otimes g\right], \tag{8}$$

where

$$\hat{P}(x) \equiv \frac{1}{8T_{\mathrm{R}} n_{\mathrm{f}} \bar{e}_q^2} \left[ x^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} - 2x \frac{\mathrm{d}}{\mathrm{d}x} + 2 \right]. \tag{9}$$

By expressing the LO term in Eq. (6) as

$$C^{(1)}_{F_{\mathrm{L}}g} \otimes g = \widetilde{F}_{\mathrm{L}} - \frac{\alpha_{\mathrm{s}}}{2\pi} C^{(2)}_{F_{\mathrm{L}}g} \otimes g \tag{10}$$

and using Eq. (8), we obtain:

$$g(x) = \hat{P}(x) \left[ \widetilde{F}_{\mathrm{L}}(x) - \frac{\alpha_{\mathrm{s}}}{2\pi} C^{(2)}_{F_{\mathrm{L}}g} \otimes g \right]. \tag{11}$$

Here, the gluon PDF in the right-hand side can be replaced with $g(x) = \hat{P}(x)\widetilde{F}_{\mathrm{L}}(x) + \mathcal{O}(\alpha_{\mathrm{s}})$. Thus, we obtain an NLO expression for the gluon PDF in terms of the structure function $F_{\mathrm{L}}$:

$$g(x) = \hat{P}(x)\widetilde{F}_{\mathrm{L}}(x) - \frac{\alpha_{\mathrm{s}}\left(Q^2\right)}{2\pi} \hat{P}(x) \left[ C^{(2)}_{F_{\mathrm{L}}g} \otimes \hat{P}\widetilde{F}_{\mathrm{L}} \right] + \mathcal{O}\left(\alpha_{\mathrm{s}}^2\right), \tag{12}$$

where we have truncated the result at $\alpha_{\mathrm{s}}^2$.

A similar iterative method can be employed to invert PDFs that include all quark flavours. The number of structure functions in the physical basis must match the number of PDFs. Moreover, the procedure here can be straightforwardly continued to higher orders in $\alpha_{\mathrm{s}}$. The perturbative truncation of the physical-basis counterparts of the PDFs implies that their numerical values do not exactly match those of the original PDFs. As a result, the momentum sum rule is not exactly satisfied in the physical basis.

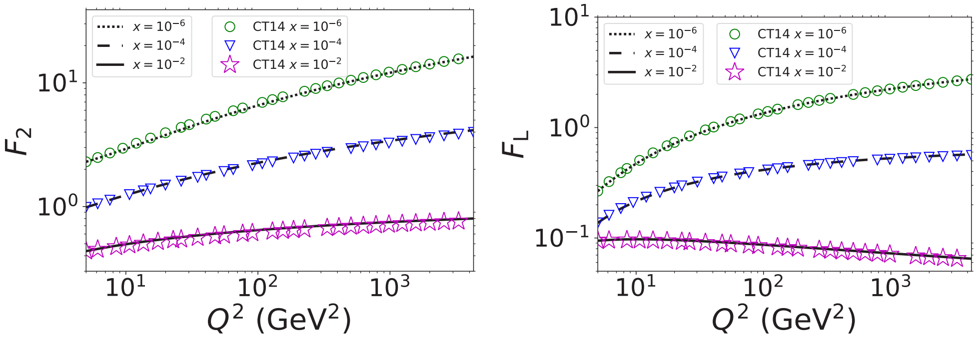
### 2.3 Results in the six-observable basis

A more complete approach to physical basis involves separating between the light quark flavours. In Ref. [11] we considered quark flavours $u$, $\overline{u}$, $d$, $\overline{d}$, and $s = \overline{s}$ together with the gluon PDF. In total we have six PDFs which then necessitates constructing a physical basis with six linearly independent structure functions. For these, we choose the structure functions $F_2$ and $F_{\mathrm{L}}$ corresponding to the virtual photon exchange, and $F_3$ corresponding to the $Z$-boson exchange. In addition, we choose the charged-current structure functions $F_3^{\mathrm{W}^-}$ and $F_{2\mathrm{c}}^{\mathrm{W}^-}$ corresponding to the $W^-$ boson exchange, and also the combination $\Delta F_2^W \equiv F_2^{\mathrm{W}^-} - F_2^{W^+}$ is included.

We implemented numerically the DGLAP evolution in the six-dimensional physical basis and compared the results to the conventional approach with PDF-based DGLAP evolution. In Fig. 1, we present the comparison of the $Q^2$ evolution of the structure functions $F_2$ and $F_{\mathrm{L}}$ in both approaches. Here, we have utilized the LHAPDF library [12] and the CT14nlo_NF3 PDF set [13]. The initial condition for the physical-basis evolution is taken from the PDF-based values. We notice a similar $Q^2$-dependence. In Fig. 2, we present the relative difference between the values corresponding to the two different approaches as a function of $x$. The differences are typically up to 10% and they increase when $x$ approaches one. The differences are due to the perturbative truncation. The uncertainties arising from the factorization scheme and scale dependencies are not shown in the Figs. 1 and 2.

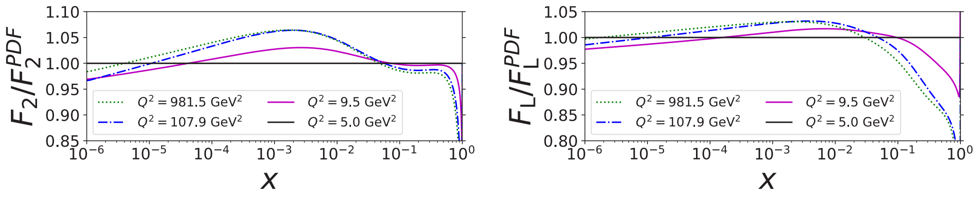
## 3 LHC cross sections in a physical basis

The physical-basis approach should, at least in principle, be a globally applicable in the same way as PDFs are. Therefore, we can express the PDF-dependent cross sections in the physical




(a) $F_2$ CT14 NLO (b) $F_{\rm L}$ CT14 NLO

Figure 1: Comparison of the structure functions $F_2$ and $F_{\rm L}$ computed from DGLAP evolution in the physical basis (black lines) to the structure functions computed from DGLAP evolved PDFs (colourful markers).




(a) $F_2/F_2^{\rm CT14}$ (b) $F_{\rm L}/F_{\rm L}^{\rm CT14}$

Figure 2: The relative differences of the structure functions $F_2$ and $F_{\rm L}$ computed in the physical basis to the corresponding structure functions computed from the PDFs.

basis – just by substituting the physical-basis counterparts for the PDFs. As an example, we consider a Higgs production by gluon fusion, for which the cross section reads:

$$\sigma(p+p\longrightarrow {\rm H}+X)=\int {\rm d}x_1{\rm d}x_2 g(x_1,\mu)g(x_2,\mu)\hat{\sigma}_{gg\to H+X}(x_1,x_2,\frac{m_H^2}{\mu_f^2}), \tag{13}$$

where $m_H$ is the Higgs mass, $g(x_1,\mu)$ and $g(x_2,\mu)$ are the gluon PDFs. Now the gluon PDFs can be expressed in the physical basis as

$$g(x,\mu_f^2)=\sum_j C_{jg}^{-1}(Q^2,\mu_f^2)\otimes F_j(Q^2), \tag{14}$$

where $F_j = F_2, F_{\rm L}/\frac{\alpha_s}{2\pi}, F_3, \Delta F_2^{\rm W}, F_3^{\rm W^-}, F_{\rm 2c}^{\rm W^-}$. By inserting this expression in Eq. (13), we obtain the Higgs production cross section in the physical basis:

$$\sigma(p+p\longrightarrow H+X)=$$
$$\int {\rm d}x_1{\rm d}x_2\hat{\sigma}_{gg\to H+X}(x_1,x_2,\frac{m_H^2}{\mu_f^2})\left[\sum_j C_{jg}^{-1}(Q^2,\mu_f^2)\otimes F_j(Q^2)\right]_{x_1}\left[\sum_k C_{kg}^{-1}(Q^2,\mu_f^2)\otimes F_k(Q^2)\right]_{x_2}. \tag{15}$$

Here the subscripts $x_1$ and $x_2$ refer to the Bjorken-$x$ values in the convolutions. According to Ref. [6] the explicit factorization scale dependence cancels when the cross sections have a similar structure as above. Hence, only ratios of physical scales remain in the logarithmic terms $\log\left(Q^2/m_H^2\right)$.

## 4 Summary and outlook

We have constructed a six-dimensional physical basis at NLO. We have made a numerical implementation for the physical-basis DGLAP evolution and compared the results to the conventional approach with PDFs. We discussed how the factorization scale and scheme dependencies do not contribute to the evolution equations in the physical basis. We demonstrated the applicability of a physical basis to LHC cross sections, using Higgs production via gluon fusion as an example process.

Future work will focus on applying the physical basis to LHC cross-section analyses. We also plan to incorporate heavy quark flavours into the framework, which will require an expanded set of structure functions to define the physical basis.

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