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Extended hydrodynamic equations for binary granular mixtures

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Abstract. The derivation of 2×14-moment equations for a dilute binary granular mixture is outlined, following Grad's moment method. In addition to the mass, momentum and energy equations for each species, the deviatoric stress tensor, the heat flux vector and the fully contracted fourth moment are treated as separate hydrodynamic fields for each species, with the last field variable is known to be important for granular gases even at the leading-order for the homogeneous cooling state of a granular gas. The expression for the non-equilibrium velocity distribution function of each species is obtained by employing an Hermite expansion in terms of fourteen hydrodynamic fields around the species Maxwellian. The production terms of each hydrodynamic equations have been calculated by including the respective quadratic-order nonlinear terms in contrast to most previous works that evaluated source terms by retaining only linear terms. The extended hydrodynamic equations are then used to analyse the homogeneous cooling state of granular mixtures.

1 Introduction

It is commonly acknowledged that the description of a gas through the infinite hierarchy of moment equations is equivalent to its description with the Boltzmann equation [1–3]. This paper presents the derivation of moment equations for a dilute binary granular mixture via Grad's moment method [1]. The primary goal here is to derive a consistent set of "beyond-Navier-Stokes" (extended) hydrodynamic equations by retaining all production terms that are second-order in hydrodynamic fields and their gradients. The homogeneous cooling state (HCS) of a monodisperse granular mixture is analysed using this second-order nonlinear theory. The usefulness of the present theory to analyse the salient features of granular gas mixtures, that appear beyond the Navier-Stokes regime (such as the non-reciprocal diffusion, the normal stress differences and various rarefaction-driven effects), is discussed in Sec. 5.

2 Boltzmann equation for a binary mixture

2.1 Dynamics of binary collision

Consider a dilute mixture of two types (labelled A and B) of smooth inelastic hard spheres of masses m_A and m_B, and diameters d_A and d_B. The inelasticity of particles is characterized by $e_{\alpha\beta}$ which denotes the coefficient of normal restitution for collisions between particles of species $\alpha \in \{A, B\}$ and $\beta \in \{A, B\}$. The relation between pre- and post-collisional velocities due to a collision of a sphere of species α with a sphere of species β [4–9] is given by

$$\left.\begin{aligned} \boldsymbol{c}'_\alpha &= \boldsymbol{c}_\alpha - (1 + e_{\alpha\beta})M^{\beta\alpha}(\boldsymbol{c}_{\alpha\beta} \cdot \hat{\boldsymbol{k}})\hat{\boldsymbol{k}} \\ \boldsymbol{c}'_\beta &= \boldsymbol{c}_\beta + (1 + e_{\alpha\beta})M^{\alpha\beta}(\boldsymbol{c}_{\alpha\beta} \cdot \hat{\boldsymbol{k}})\hat{\boldsymbol{k}} \end{aligned}\right\}, \tag{1}$$

where $\{\boldsymbol{c}_\alpha, \boldsymbol{c}_\beta\}$ denote the pre-collisional velocities of the two colliding spheres, and $\{\boldsymbol{c}'_\alpha, \boldsymbol{c}'_\beta\}$ are their post-collisional velocities; $\hat{\boldsymbol{k}}$ is the unit contact vector between the centers of two colliding spheres,

$$M^{\alpha\beta} \equiv \frac{m_\alpha}{m_\alpha + m_\beta} \quad \text{and} \quad \boldsymbol{c}_{\alpha\beta} \equiv \boldsymbol{c}_\alpha - \boldsymbol{c}_\beta. \tag{2}$$

2.2 Boltzmann equation

The description of a binary granular mixture consisting of two monatomic-inert-ideal gases in the phase space is provided by the two velocity distribution functions of individual gases of the mixture:

$$f_\alpha \equiv f_\alpha(\boldsymbol{x}, \boldsymbol{c}_\alpha, t) \;\; \text{for} \;\; \alpha = A, B. \tag{3}$$

Here, $\boldsymbol{x} \equiv (x, y, z)^T$, $\boldsymbol{c}_\alpha \equiv (c_x^{(\alpha)}, c_y^{(\alpha)}, c_z^{(\alpha)})^T$ and t denote the position, instantaneous velocity of the species α and time, respectively. The velocity distribution function $f_\alpha(\boldsymbol{x}, \boldsymbol{c}_\alpha, t)$ is defined such that $f_\alpha(\boldsymbol{x}, \boldsymbol{c}_\alpha, t)d\boldsymbol{x}d\boldsymbol{c}_\alpha$ gives the number of particles of the species α at time t in a differential volume $d\boldsymbol{x}$ centred around the position $\boldsymbol{x}$ and differential volume $d\boldsymbol{c}_\alpha$ in velocity space located around $\boldsymbol{c}_\alpha$.

The velocity distribution functions of a binary mixture satisfy two Boltzmann equations [4, 5, 7, 8]

$$\frac{\partial f_\alpha}{\partial t} + c_i^{(\alpha)}\frac{\partial f_\alpha}{\partial x_i} = \sum_{\beta=A,B} \mathfrak{B}_{\alpha\beta}(f_\alpha, f_\beta, e_{\alpha\beta}), \quad \alpha = A, B \tag{4}$$

where the species collision operator is defined by:

$$\mathfrak{B}_{\alpha\beta} = d_{\alpha\beta}^2 \int\int_{\boldsymbol{c}_{\alpha\beta}\cdot\hat{\boldsymbol{k}}>0} \left[\frac{f_\alpha(\boldsymbol{c}'_\alpha)f_\beta(\boldsymbol{c}'_\beta)}{e_{\alpha\beta}^2} - f_\alpha(\boldsymbol{c}_\alpha)f_\beta(\boldsymbol{c}_\beta)\right](\boldsymbol{c}_{\alpha\beta} \cdot \hat{\boldsymbol{k}})d\boldsymbol{c}_\beta d\hat{\boldsymbol{k}}. \tag{5}$$

with $d_{\alpha\beta} = (d_\alpha + d_\beta)/2$.

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3 Extended hydrodynamic theory

For an arbitrary function $\Psi_\alpha \equiv \Psi_\alpha(\boldsymbol{x}, \boldsymbol{c}_\alpha, t)$, its average $\langle\Psi_\alpha\rangle$ is defined in terms of the velocity distribution function $f_\alpha(\boldsymbol{x}, \boldsymbol{c}_\alpha, t)$ as

$$n_\alpha\langle\Psi_\alpha\rangle = \int \Psi_\alpha f_\alpha(\boldsymbol{x}, \boldsymbol{c}_\alpha, t) d\boldsymbol{c}_\alpha. \tag{6}$$

The number density $n_\alpha(\boldsymbol{x}, t)$, mass density $\rho_\alpha(\boldsymbol{x}, t)$, macroscopic velocity $\boldsymbol{u}_\alpha(\boldsymbol{x}, t)$, granular temperature $T_\alpha(\boldsymbol{x}, t)$, stress tensor $\boldsymbol{\sigma}_\alpha(\boldsymbol{x}, t)$, heat flux vector $\boldsymbol{q}_\alpha(\boldsymbol{x}, t)$ and contracted fourth moment $p^{(\alpha)}_{iijj}(\boldsymbol{x}, t)$ of α-constituent are expressed in terms of moments by choosing Ψ_α in Eq. (6):

$$\left.\begin{array}{l} \rho_\alpha = m_\alpha \int f_\alpha d\boldsymbol{c}_\alpha = m_\alpha n_\alpha, \quad n_\alpha \boldsymbol{u}_\alpha = \int \boldsymbol{c}_\alpha f_\alpha d\boldsymbol{c}_\alpha \\ \frac{3}{2} n_\alpha T_\alpha = \frac{1}{2} m_\alpha \int C_\alpha^2 f_\alpha d\boldsymbol{c}_\alpha \\ \sigma^{(\alpha)}_{ij} = m_\alpha \int C^{(\alpha)}_{\langle i} C^{(\alpha)}_{j\rangle} f_\alpha d\boldsymbol{c}_\alpha \\ q^{(\alpha)}_i = \frac{1}{2} m_\alpha \int C_\alpha^2 C^{(\alpha)}_i f_\alpha d\boldsymbol{c}_\alpha \\ p^{(\alpha)}_{iijj} = m_\alpha \int C_\alpha^4 f_\alpha d\boldsymbol{c}_\alpha \end{array}\right\}, \tag{7}$$

where

$$\boldsymbol{C}_\alpha(\boldsymbol{x}, \boldsymbol{c}_\alpha, t) = \boldsymbol{c}_\alpha - \boldsymbol{u}(\boldsymbol{x}, t) \tag{8}$$

is the peculiar velocity of the α-species with respect to the mixture velocity of the binary gas mixture $\boldsymbol{u}(\boldsymbol{x}, t)$ as defined in Eq. (10), and the angular brackets around the indices indicate the symmetric and trace-free tensor.

Denoting the total density, total pressure, total stress tensor, total heat flux and total contracted fourth moment for the binary mixture by $\rho, p, \boldsymbol{\sigma}, \boldsymbol{q}$ and p_{iijj}, respectively, and the granular temperature for the mixture by T, we have

$$\left.\begin{array}{l} \rho = \sum_{\alpha=A}^B \rho_\alpha, \quad p = nT = \sum_{\alpha=A}^B n_\alpha T_\alpha \\ \sigma_{ij} = \sum_{\alpha=A}^B \sigma^{(\alpha)}_{ij}, \quad q_i = \sum_{\alpha=A}^B q^{(\alpha)}_i \\ p_{iijj} = \sum_{\alpha=A}^B p^{(\alpha)}_{iijj}, \quad n = \sum_{\alpha=A}^B n_\alpha \end{array}\right\}. \tag{9}$$

The mixture velocity (or, the mass average velocity) $\boldsymbol{u}(\boldsymbol{x}, t)$ is defined as the ratio of total momentum density to total density

$$\boldsymbol{u}(\boldsymbol{x}, t) = \frac{1}{\rho} \sum_{\alpha=A}^B \rho_\alpha \boldsymbol{u}_\alpha = \frac{\rho_A \boldsymbol{u}_A + \rho_B \boldsymbol{u}_B}{\rho_A + \rho_B}. \tag{10}$$

In general, the mixture velocity $\boldsymbol{u}(\boldsymbol{x}, t)$ is different from the macroscopic velocity of each species $\boldsymbol{u}_\alpha$ which leads to a phenomenon called species diffusion. The latter is described by the diffusion velocity of each species via

$$\boldsymbol{v}_\alpha = \boldsymbol{u}_\alpha - \boldsymbol{u}, \tag{11}$$

which satisfies the compatibility condition

$$\sum_{\alpha=A}^B \rho_\alpha \boldsymbol{v}_\alpha = \rho_A \boldsymbol{v}_A + \rho_B \boldsymbol{v}_B = \boldsymbol{0}, \tag{12}$$

that follows from Eq. (10).

3.1 Moment equations

To derive moment equations for the species α, we multiply the Boltzmann equation (4) by an arbitrary function $\Psi_\alpha(\mathbf{c}_\alpha)$ and integrate it over the velocity space. The 14-moment equations for species α corresponding to 14 field variables $\{\rho_\alpha, u^{(\alpha)}_i, T_\alpha, \sigma^{(\alpha)}_{ij}, q^{(\alpha)}_i, p^{(\alpha)}_{iijj}\}$ are then obtained by replacing Ψ_α in the master balance equation with each element of $\Psi_\alpha = m_\alpha\{1, c^{(\alpha)}_i, C_\alpha^2, C^{(\alpha)}_{\langle i} C^{(\alpha)}_{j\rangle}, C_\alpha^2 C^{(\alpha)}_i, C_\alpha^4\}$.

The first 14 moment equations for the species α are

$$\left.\begin{array}{l} \frac{\partial \rho_\alpha}{\partial t} + \frac{\partial}{\partial x_k}(\rho_\alpha u^{(\alpha)}_k) = S^{(\alpha)}_1, \\ \frac{\partial}{\partial t}(\rho_\alpha u^{(\alpha)}_i) + \frac{\partial}{\partial x_k}(P^{(\alpha)}_{ik} + \rho_\alpha(u_i u^{(\alpha)}_k + u_k u^{(\alpha)}_i - u_i u_k)) = S^{(\alpha)}_2(i), \\ \frac{\partial}{\partial t}(3nT) + \frac{\partial}{\partial x_k}(2q^{(\alpha)}_k + 3u_k nT) + 2\frac{Du_i}{Dt}(\rho_\alpha u^{(\alpha)}_i - \rho_\alpha u_i) + 2\frac{\partial u_i}{\partial x_k} P^{(\alpha)}_{ik} = S^{(\alpha)}_3, \\ \frac{\partial}{\partial t}(\sigma^{(\alpha)}_{ij}) + \frac{\partial}{\partial x_k}(Q^{(\alpha)}_{ijk} + u_k P^{(\alpha)}_{ij} - \frac{1}{3}\delta_{ij}(2q^{(\alpha)}_k + 3u_k n_\alpha T_\alpha)) + \frac{Du_j}{Dt}(\rho_\alpha u^{(\alpha)}_i - \rho_\alpha u_i) + \frac{Du_i}{Dt}(\rho_\alpha u^{(\alpha)}_j - \rho_\alpha u_j) \\ \qquad -\frac{2}{3}\delta_{ij}\frac{Du_m}{Dt}(\rho_\alpha u^{(\alpha)}_m - \rho_\alpha u_m) + \frac{\partial u_i}{\partial x_k} P^{(\alpha)}_{jk} + \frac{\partial u_j}{\partial x_k} P^{(\alpha)}_{ik} - \frac{2}{3}\delta_{ij}\frac{\partial u_m}{\partial x_k} P^{(\alpha)}_{km} = S^{(\alpha)}_4(ij), \\ \frac{\partial}{\partial t}(2q^{(\alpha)}_i) + \frac{\partial}{\partial x_k}(R^{(\alpha)}_{rrik} + 2u_k q^{(\alpha)}_i) + \frac{Du_i}{Dt}(3n_\alpha T_\alpha) + 2\frac{Du_m}{Dt} P^{(\alpha)}_{im} + 2\frac{\partial u_i}{\partial x_k} q^{(\alpha)}_k + 2\frac{\partial u_m}{\partial x_k} Q^{(\alpha)}_{kim} = S^{(\alpha)}_5(i), \\ \frac{\partial}{\partial t}(p^{(\alpha)}_{iijj}) + \frac{\partial}{\partial x_k}(p^{(\alpha)}_{iijjk} + u_k p^{(\alpha)}_{iijj}) + 8\frac{Du_i}{Dt} q^{(\alpha)}_i + 4\frac{\partial u_i}{\partial x_k} R^{(\alpha)}_{rrik} = S^{(\alpha)}_6, \end{array}\right\}, \tag{13}$$

where

$$\left.\begin{array}{l} P^{(\alpha)}_{ij} = m_\alpha \int f_\alpha C^{(\alpha)}_i C^{(\alpha)}_j d\boldsymbol{c}_\alpha \\ Q^{(\alpha)}_{ijk} = m_\alpha \int f_\alpha C^{(\alpha)}_i C^{(\alpha)}_j C^{(\alpha)}_k d\boldsymbol{c}_\alpha \\ R^{(\alpha)}_{rrik} = m_\alpha \int f_\alpha C_\alpha^2 C^{(\alpha)}_i C^{(\alpha)}_k d\boldsymbol{c}_\alpha \\ p^{(\alpha)}_{iijjk} = m_\alpha \int f_\alpha C_\alpha^4 C^{(\alpha)}_k d\boldsymbol{c}_\alpha \end{array}\right\}. \tag{14}$$

The integral expressions for production/source terms $S^{(\alpha)}_i$ in Eqs. (13) are omitted for the sake of brevity; the determination of $S^{(\alpha)}_i$ requires an expression for the non-equilibrium distribution function which is considered in Sec. 3.2.

3.2 Grad's distribution function for species α

In order to determine the velocity distribution function for species α, one can expand the distribution function f_α in an infinite series with the Hermite polynomials $H^{(\alpha)}_{i_1 i_2 \ldots i_N}(N = 0, 1, 2, \cdots)$ [9, 10] being the basis functions:

$$f_\alpha = f^{(\alpha)}_M (a^{(\alpha)} H^{(\alpha)} + a^{(\alpha)}_i H^{(\alpha)}_i + \cdots + \frac{1}{N!} a^{(\alpha)}_{i_1 i_2 \cdots i_N} H^{(\alpha)}_{i_1 i_2 \cdots i_N} + \cdots), \tag{15}$$

where $a^{(\alpha)}_{i_1 i_2 \cdots i_N}(N = 0, 1, 2, \cdots)$ are coefficients that depend on $(\boldsymbol{x}, t)$ and

$$f^{(\alpha)}_M = n_\alpha \left(\frac{\beta_\alpha}{\pi}\right)^{3/2} \exp(-\beta_\alpha C_\alpha^2) = n_\alpha \left(\frac{m_\alpha}{T_\alpha}\right)^{3/2} \omega(\boldsymbol{\xi}^{(\alpha)}), \tag{16}$$

where $\beta_\alpha = m_\alpha/2T_\alpha$ and the weight function $\omega(\boldsymbol{\xi}^{(\alpha)})$ is

$$\omega(\boldsymbol{\xi}^{(\alpha)}) = \frac{1}{(2\pi)^{3/2}} \exp\left(-\frac{\xi^{(\alpha)2}}{2}\right), \text{ with } \xi_i^{(\alpha)} = \sqrt{\frac{m_\alpha}{T_\alpha}} C_i^{(\alpha)}. \tag{17}$$

Substituting the distribution function (15) into the definitions of the moments of the distribution function and using the orthogonality relations for Hermite polynomials, we obtain the 14-moment Grad's distribution function

$$\begin{aligned} f_\alpha &= f_M^{(\alpha)}\Big[1 + 2\beta_\alpha(u_i^{(\alpha)} - u_i)C_i^{(\alpha)}\Big(\frac{7}{2} - \beta_\alpha C_\alpha^2\Big) \\ &+ \frac{2\beta_\alpha^2}{\rho_\alpha}\sigma_{ij}^{(\alpha)} C_i^{(\alpha)} C_j^{(\alpha)} + \frac{8}{5}\frac{\beta_\alpha^2}{\rho_\alpha} q_i^{(\alpha)} C_i^{(\alpha)}\Big(\beta_\alpha C_\alpha^2 - \frac{5}{2}\Big) \\ &+\Delta_\alpha\Big(\frac{\beta_\alpha^2 C_\alpha^4}{2} - \frac{5\beta_\alpha C_\alpha^2}{2} + \frac{15}{8}\Big)\Big], \end{aligned} \tag{18}$$

where

$$\Delta_\alpha = \frac{p_{iijj}^{(\alpha)} - p_{iijj}^{M(\alpha)}}{p_{iijj}^{M(\alpha)}} \tag{19}$$

is the dimensionless non-equilibrium part of the contracted fourth-order moment and

$$p_{iijj}^{M(\alpha)} = m_\alpha \int C_\alpha^4 f_M^{(\alpha)} d\boldsymbol{c}_\alpha = \rho_\alpha\left(\frac{15}{4\beta_\alpha^2}\right) \tag{20}$$

is the contracted fourth-order moment evaluated for a Maxwellian distribution function.

3.3 Linear production terms

Using Grad's distribution function (18), we can determine the full linear and non-linear production terms $S_i^{(\alpha)}$ (that appear on the right-hand side of Eqs. (13)). The production terms that are linear in hydrodynamic fields are given by:

$$\left.\begin{aligned} &S_1^{(\alpha)} = 0, \\ &S_2^{(\alpha)}(i) = \textstyle\sum_{\beta=A}^{B} K_1 q_i^{(\beta)} + K_2 q_i^{(\alpha)} + K_3(u_i^{(\alpha)} - u_i) \\ &\qquad\qquad + K_4(u_i^{(\beta)} - u_i), \\ &S_3^{(\alpha)} = \textstyle\sum_{\beta=A}^{B} L_1 + L_2\Delta^{(\alpha)} + L_3\Delta^{(\beta)}, \\ &S_4^{(\alpha)}(ij) = \textstyle\sum_{\beta=A}^{B} M_1\sigma_{ij}^{(\alpha)} + M_2\sigma_{ij}^{(\beta)}, \\ &S_5^{(\alpha)}(i) = \textstyle\sum_{\beta=A}^{B} N_1 q_i^{(\beta)} + N_2 q_i^{(\alpha)} + N_3(u_i^{(\alpha)} - u_i) \\ &\qquad\qquad + N_4(u_i^{(\beta)} - u_i), \\ &S_6^{(\alpha)} = \textstyle\sum_{\beta=A}^{B} Q_1 + Q_2\Delta^{(\alpha)} + Q_3\Delta^{(\beta)}, \end{aligned}\right\}, \tag{21}$$

and the expressions for K_i, L_i, M_i, N_i, Q_i's are omitted that can be found in the PhD thesis of the first author (Rongali 2017, JNCASR).

3.4 Non-linear production terms

The non-zero production terms of quadratic order are given by:

$$\begin{aligned} S_2^{(\alpha)} &= \sum_{\beta=A}^{B} C_1\Delta^{(\alpha)} q_i^{(\beta)} + C_2\Delta^{(\beta)} q_i^{(\alpha)} + C_3\Delta^{(\beta)} J_i^{(\alpha)} \\ &+ C_4\Delta^{(\alpha)} J_i^{(\beta)} + C_5 q_k^{(\beta)}\sigma_{ik}^{(\alpha)} + C_6 J_k^{(\beta)}\sigma_{ik}^{(\alpha)} \\ &+ C_7 q_k^{(\alpha)}\sigma_{ik}^{(\beta)} + C_8 J_k^{(\alpha)}\sigma_{ik}^{(\beta)}, \end{aligned} \tag{22}$$

$$\begin{aligned} S_3^{(\alpha)}(i) &= \sum_{\beta=A}^{B} A_1 q_i^{(\alpha)} q_i^{(\beta)} + A_3 q_i^{(\alpha)} J_i^{(\beta)} + A_2 q_k^{(\alpha)} q_k^{(\beta)} \\ &+ A_4 q_k^{(\alpha)} J_k^{(\beta)} + A_5 q_k^{(\beta)} J_k^{(\alpha)} + A_6 J_i^{(\alpha)} J_i^{(\beta)} \\ &+ A_8\Delta^{(\alpha)}\Delta^{(\beta)} + A_7 J_k^{(\alpha)} J_k^{(\beta)} + A_9\sigma_{ij}^{(\alpha)}\sigma_{ij}^{(\beta)} \\ &+ A_{10}\sigma_{ik}^{(\alpha)}\sigma_{ik}^{(\beta)} + A_{11} q_i^{(\beta)} J_i^{(\alpha)}, \end{aligned} \tag{23}$$

$$\begin{aligned} S_4^{(\alpha)}(ij) &= \sum_{\beta=A}^{B} D_1 J_i^{(\alpha)} q_j^{(\beta)} + D_2 J_i^{(\alpha)} J_j^{(\beta)} + D_3 q_i^{(\alpha)} J_j^{(\beta)} \\ &+ D_4 q_i^{(\alpha)} q_j^{(\beta)} + D_5 q_i^{(\beta)} J_j^{(\alpha)} + D_6 J_i^{(\beta)} J_j^{(\alpha)} \\ &+ D_7 J_i^{(\beta)} q_j^{(\alpha)} + D_8 q_i^{(\beta)} q_j^{(\alpha)} + D_9\Delta^{(\beta)}\sigma_{ij}^{(\alpha)} \\ &+ D_{10}\Delta^{(\alpha)}\sigma_{ij}^{(\beta)} + D_{11}\sigma_{jk}^{(\alpha)}\sigma_{ik}^{(\beta)} + D_{12}\sigma_{ik}^{(\alpha)}\sigma_{jk}^{(\beta)} \\ &+ \delta_{ij}(D_{13} J_i^{(\alpha)} q_i^{(\beta)} + D_{14} J_i^{(\alpha)} J_i^{(\beta)} + D_{15} q_i^{(\alpha)} J_i^{(\beta)} \\ &+ D_{16} q_i^{(\alpha)} q_i^{(\beta)} + D_{17}\sigma_{ik}^{(\alpha)}\sigma_{ik}^{(\beta)}), \end{aligned} \tag{24}$$

$$\begin{aligned} S_5^{(\alpha)}(i) &= \sum_{\beta=A}^{B} E_1\Delta^{(\beta)} J_i^{(\alpha)} + E_2\Delta^{(\beta)} q_i^{(\alpha)} + E_3\Delta^{(\alpha)} q_i^{(\beta)} \\ &+E_4\Delta^{(\alpha)} J_i^{(\beta)} + E_5 q_k^{(\beta)}\sigma_{ik}^{(\alpha)} + E_6 J_k^{(\beta)}\sigma_{ik}^{(\alpha)} \\ &+E_7 q_k^{(\alpha)}\sigma_{ik}^{(\beta)} + E_8 J_k^{(\alpha)}\sigma_{ik}^{(\beta)}, \end{aligned} \tag{25}$$

$$\begin{aligned} S_6^{(\alpha)} &= \sum_{\beta=A}^{B} B_2 q_k^{(\alpha)} q_k^{(\beta)} + B_4 q_k^{(\alpha)} J_k^{(\beta)} + B_5 q_k^{(\beta)} J_k^{(\alpha)} \\ &+ B_7 J_k^{(\alpha)} J_k^{(\beta)} + B_8\Delta^{(\alpha)}\Delta^{(\beta)} + B_9\sigma_{ij}^{(\alpha)}\sigma_{ij}^{(\beta)}. \end{aligned} \tag{26}$$

The expressions for A_i, B_i, C_i, D_i, E_i's and the definition of $J_k^{(\alpha)}$ can be found in the PhD thesis of the first author (Rongali 2017, JNCASR).

4 Homogeneous cooling state

Here we analyse the homogeneous cooling state of the limiting case of a mono-disperse granular mixture. One can obtain the production terms for a monodisperse gas by ignoring the summation and replacing β with α in equations (21) and substitute $m_A = m_B = m$, $n_A = n_B = n$, $d_{AA} = d_{BB} = d_{AB} = d$, $e_{AA} = e_{BB} = e_{AB} = e$, $\sigma_{ij}^{(A)} = \sigma_{ij}^{(B)} = \sigma_{ij}$, $q_i^{(A)} = q_i^{(B)} = q_i$ and $\Delta_A = \Delta_B = \Delta$. The collisional production/source terms for the monodisperse case become

$$\left.\begin{aligned} S_1 &= S_1^{(M)} = 0, \quad S_2 = S_2^{(M)} = 0 \\ S_3 &= S_3^{(M)} = -\frac{4d^2}{m}\sqrt{\pi}(1-e^2)\left(1+\frac{3\Delta}{16}\right)\rho^2\left(\frac{T}{m}\right)^{3/2} \\ S_4 &= S_4^{(M)} = -\frac{4}{5}\frac{d^2}{m}\sqrt{\pi}(1+e)(3-e)\rho\left(\frac{T}{m}\right)^{1/2}\sigma_{ij} \\ S_5 &= S_5^{(M)} = -\frac{2}{15}\frac{d^2}{m}\sqrt{\pi}(1+e)(49-33e)\rho\left(\frac{T}{m}\right)^{1/2}q_i \\ S_6 &= S_6^{(M)} = -\frac{4d^2}{m}\sqrt{\pi}(1+e)\Big[(2e^2+9)(1-e) \\ &\quad +(271-207e+30e^2-30e^3)\frac{\Delta}{16}\Big]\rho^2\left(\frac{T}{m}\right)^{5/2} \end{aligned}\right\}. \tag{27}$$

The above production terms for mono-disperse case are identical to those in Refs. [11, 12].

The moment equation of contracted fourth moment (Δ) for a spatially homogeneous system becomes

$$15\rho\left(\frac{T}{m}\right)^2\frac{d}{dt}(\Delta) = S_6^{(M)} - 10S_3^{(M)}(1+\Delta)\left(\frac{T}{m}\right). \tag{28}$$

For the steady-state contracted fourth moment ($\Delta^\infty = \lim_{t\to\infty}\Delta$), Eq. (28) simplifies to

$$S_6^{(M)} - 10S_3^{(M)}(1+\Delta^\infty)\left(\frac{T}{m}\right) = 0. \tag{29}$$

Substituting the expressions for $S_6^{(M)}$ and $S_3^{(M)}$ in the above equation and neglecting the products of Δ^∞, we obtain

$$\Delta^\infty = \frac{16(1-2e^2)(1-e)}{81-17e+30e^2(1-e)}. \tag{30}$$

This expression for Δ^∞ agrees with that of Ref. [13]. Note that Δ^∞ is negative for $e \sim 1$ but becomes positive for highly dissipative particles ($e < 0.8$).

5 Summary and outlook

The derivation of the 2×14-moment equations for a dilute binary granular mixture is presented. To close the hydrodynamic equations, all production terms have been calculated by using the 14-moment distribution function as functions of (i) the restitution coefficient ($e_{\alpha\beta}$), (ii) the mass and size ratios of two species and (iii) the number density ratio of each species. In the mono-disperse limit, the calculated "linear" and "nonlinear" production terms agree with previous works.

The applicability of the present extended hydrodynamic equations can be assessed by evaluating various Burnett-order effects [2, 14–19]. For example, the normal stress differences [14, 15, 20] in a sheared system of binary granular mixtures can be evaluated based on the present theory and the theoretical predictions can be tested against particle-level simulations [21–23]. The phenomenon of anisotropic diffusion, which is absent in Navier-Stokes-level hydrodynamic theory, can be analysed with the present nonlinear theory. Also, to apply these extended hydrodynamic equations to "wall-bounded" granular flows such as in gravity-driven Poiseuille flow (i.e. vertical chute flow) [24–28], we need to derive boundary conditions on higher-order fields – this can be done by assuming Maxwell-type accommodation models for inelastic particle-wall collisions [29, 30].

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