Harmonic resonances in granular plane Couette flow

Lima Biswas[1,*] and *Priyanka* Shukla[2,**]

[1]Department of Mathematics, Gandhi Institute of Technology and Management, Hyderabad 502329, India
[2]Department of Mathematics, Indian Institute of Technology Madras, Chennai 600036, India

Abstract. The shearbanding instability patterns and bifurcations have been well-studied in granular plane Couette flow using the single-mode Landau equations. However, shearbanding modes interact with each other and undergo harmonic resonances. The present study explores the nonlinear resonant interactions among these shearbanding modes. Using analytical solutions of the linear stability analysis, we determine the critical parameters for harmonic resonance, particularly the 1:2 resonance. At these resonant points, the single-mode Landau equation is invalid because it does not capture the interactions. Thus, a coupled Landau equation is derived to understand 1:2 resonance, where one wavenumber acts as the fundamental harmonic of another. It is found that there is energy exchange between the primary and second-harmonic shear bands, together with amplitude modulation. Thus, harmonic resonance substantially affects macroscopic flow behavior by enhancing or decaying energy dissipation.

1 Introduction

Granular Couette flow (GCF) describes the shear-induced movement and deformation of collections of discrete particles, such as sand, grains, or powders. These flows exhibit complex rheology, transitioning between solid-like resistance and fluid-like motion based on applied forces and particle interactions. Unlike conventional fluid particles, granular particles can form force chains, experience jamming, or undergo sudden flow transitions, making their behavior highly nonlinear and dependent on factors such as particle size, shape, and friction. These flows are found in both natural and industrial settings, and play a crucial role in processes like mineral processing, pharmaceuticals, and food production industries, where controlling flow dynamics is essential for efficiency and quality [1].

The behavior of GCFs is categorized into distinct regimes based on stress conditions and particle interactions. These regimes are: (i) the quasi-static regime, where enduring force chains dominate and the flow is well-described by soil mechanics principles; (ii) the collisional regime, in which particle collisions drive the dynamics resembling the behavior of granular gases; and (iii) dense flow regime, where both frictional contacts and collisions influence the flow. GCFs in the collisional regime often exhibit various dynamical phenomena. One such dynamical behavior is the emergence of 1:2 harmonic resonances, where periodic energy exchanges between different modes of motion create structured patterns within the flow [2]. These resonances arise from particle collisions, forming self-organized states that impact transport properties and energy dissipation. Additionally, harmonic resonances can stabilize, destabilize, or restructure shear bands in GCFs, influencing overall flow behavior. In this context, the present study focuses on a specific harmonic resonance, namely the 1:2 resonance, in uniform GCF. Specifically, parameters are identified for the 1:2 resonance. Further, the coupled amplitude equations are derived and equilibrium solutions are analyzed.

2 Problem formulation

Consider a plane Couette flow of monodisperse, smooth, inelastic granular particles between two infinite horizontal plates separated by a distance h. The particles are assumed to be monodisperse, smooth, inelastic hard disks with restitution coefficient e, diameter d_p, and mass density $\bar{\rho} = \rho_p \phi$, with ρ_p and ϕ being the mass density of each particle and volume fraction, respectively. The flow is driven by the top and bottom plates moving in opposite directions along the stream-wise (x) direction, with speed $\bar{U}_w/2$. Here, our focus is on studying shearbanding along the flow gradient direction. The balance equations are thus considered to be stream-wise independent, i.e., the flow variables depend only on y. Using scales for length, velocity, time, and mass density as h, $\bar{U}_w$, $h/\bar{U}_w$, and ρ_p, respectively, the non-dimensional mass, momentum and energy equations read

$$\frac{\partial \phi}{\partial t} + v\frac{\partial \phi}{\partial y} + \phi\frac{\partial v}{\partial y} = 0,\ \phi\left(\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial y}\right) = \frac{1}{H^2}\frac{\partial}{\partial y}\left(\eta\frac{\partial u}{\partial y}\right), \tag{1a}$$

$$\phi\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial y}\right) = \frac{1}{H^2}\left(\frac{-\partial p}{\partial y} + \frac{\partial}{\partial y}\left(\lambda\frac{\partial v}{\partial y}\right) + 2\frac{\partial}{\partial y}\left(\eta\frac{\partial v}{\partial y}\right)\right), \tag{1b}$$

$$\phi\left(\frac{\partial T}{\partial t} + v\frac{\partial T}{\partial y}\right) = \frac{1}{H^2}\frac{\partial}{\partial y}\left(\kappa\frac{\partial T}{\partial y} + \mu\frac{\partial \phi}{\partial y}\right) - p\frac{\partial v}{\partial y} + \lambda\left(\frac{\partial v}{\partial y}\right)^2 + \eta\left(\frac{\partial u}{\partial y}\right)^2 + 2\eta\left(\frac{\partial v}{\partial y}\right)^2 - \mathcal{D}_0 - \mathcal{D}_u\frac{\partial v}{\partial y}, \tag{1c}$$

*e-mail: lbiswas@gitam.edu
**e-mail: priyanka@iitm.ac.in

where (u, v), ϕ and T are the velocity vector, volume fraction and granular energy, respectively, and $H = h/d$ is the scaled Couette gap. Other parameters p, η, λ, κ, μ, $\mathcal{D}_0$, $\mathcal{D}_u$ are the pressure, shear viscosity, bulk viscosity, pseudothermal conductivity, Dufour-like coefficient, zeroth- and first-order contributions to the collisional dissipation, respectively, and given in [3]. The boundary conditions at $y = \pm 1/2$ are $u = \pm 1/2$ and $\partial T/\partial y = 0$.

3 Linear stability analysis (LSA)

The base flow under study is steady, fully developed, two-dimensional flow, i.e. $\phi = \phi(y)$, $u = u(y)$, $v = 0$, $T = T(y)$. Under the above assumption, the base state is composed of a linear velocity ($u = y$) with a constant volume fraction ($\phi = \phi_0$) and constant granular temperature ($T = T_0$). To derive the linear stability equations, we decompose each flow variable as the base flow plus infinitesimal perturbation, substitute the decomposition into (1a)–(1c), and then subtract the base flow equations, and then collecting only the linear terms of the perturbation variables, we get the linearized form of disturbance equations, which can be written in matrix form as $\partial \boldsymbol{X}'/\partial t = \mathcal{L}\boldsymbol{X}'$, where the vector $\boldsymbol{X}' = (\phi', u', v', T')^{\mathsf{T}}$ represents the perturbed flow variables and $\mathcal{L}$ is the linear stability operator associated with the linearized equations. The boundary conditions are $u'(y = \pm 1/2) = \partial T'/\partial y|_{y=\pm 1/2} = 0$

Furthermore, assume a normal mode solution, i.e. $\boldsymbol{X}(y,t) = \hat{X}(y)\exp(\omega t)$, $\hat{X} = (\hat{\phi}, \hat{u}, \hat{v}, \hat{T})^{\mathsf{T}}$, $\omega = \omega_r + \mathrm{i}\,\omega_i$ is the complex frequency whose real and imaginary parts represent the growth or decay rate and frequency of the perturbations, respectively. Substituting normal mode solution in linearized disturbance equation, we get

$$\hat{L}\hat{X} = \omega\hat{X}, \qquad \hat{u}\,(\pm 1/2) = \mathrm{d}\hat{T}/\mathrm{d}y = 0, \qquad (2)$$

where $\hat{L} = \mathcal{L}\left(\partial/\partial y \to \mathrm{d}/\mathrm{d}y, \partial^2/\partial y^2 \to \mathrm{d}^2/\mathrm{d}y^2\right)$. This system admits an analytical solution, which is given as, $\hat{X}(t,y) = (\phi_1(t)\cos\Gamma, u_1(t)\sin\Gamma, v_1(t)\sin\Gamma, T_1(t)\sin\Gamma)$ where $\Gamma = \pi\beta(y \pm 1/2)$, and $\beta = 1, 2, 3, \ldots$ is the mode number and $(\phi_1, u_1, v_1, T_1)^{\mathsf{T}}$ is the amplitude of the normal mode solution [4]. Applying analytical solutions in (2), we get $L_1X_1 = \omega X_1$, where $X_1 = (\phi_1, u_1, v_1, T_1)^{\mathsf{T}}$ and L_1 is simplified form of $\hat{L}$. For more details on LSA, see [3]. Figure 1 presents the neutral stability curves for mode numbers $\beta = 1, 2, 3$ and for restitution coefficient $e = 0.6$. The flow is unstable (stable) inside (outside) the contour. There is a critical volume fraction $\phi^0 = \phi^0_{crit} = 0.3926$ below which the flow is linearly stable for all mode numbers. Similarly, there is a critical Couette gap H_{crit} below which the flow is always linearly stable. However, H_{crit} increases with the mode number, indicating that the onset of linear instability decreases with the mode number. Note that uniform shear flow is subcritically unstable [5, 6]. We look for the occurrence of 1:2 resonant interactions, where the flow is linearly unstable, i.e., supercritical region.

4 Existence of 1:2 resonance interaction

We first focus on identifying the harmonic resonance, specifically the 1:2 resonance in uniform GCF, where the fundamental shearbanding mode interacts with its second harmonic. Specifically, the two frequencies or the eigenvalues (say ω_1 and ω_2) and the corresponding mode numbers (say β_1 and β_2) of the interacting modes satisfy the relation $\omega_1 : \omega_2 = \beta_1 : \beta_2$ =1:2. Note that, in uniform GCF eigenvalues are always real. Figure 2 depicts the variation of ω^l/β with the scaled Couette gap H for various mode numbers. At each intersection point (circle) the ratio between the growth rates and the ratio between the corresponding mode number are equal to 1:2, which is the required condition for the 1:2 resonant interaction. At the resonance points the single amplitude equation becomes invalid, and we need to study the coupled amplitude equations in order to understand energy transfer mechanism and bifurcations in shearbanding at resonant parameters [6]. Hereafter β_1 and β_2 will be used to denote the mode numbers of the 1:2 interacting modes; and ω^l_1 and ω^l_2 will be used for the corresponding growth rates. Table 1 displays parameters where 1:2 resonance exists.

From linear stability analysis, it follows that if the mode number and frequency pair (β, ω) satisfies (2) at $H = H_1$, then the pairs $(2\beta, \omega)$, $(3\beta, \omega)$, and so on, also satisfy equation (2) at $H = 2H_1$, $H = 3H_1$, respectively. This indicates a systematic scaling behavior of the modes with increasing H. This suggests that if (β_1, ω_1) and (β_2, ω_2) satisfy the resonance conditions $\omega_1 : \omega_2 = \beta_1 : \beta_2$ =1:2

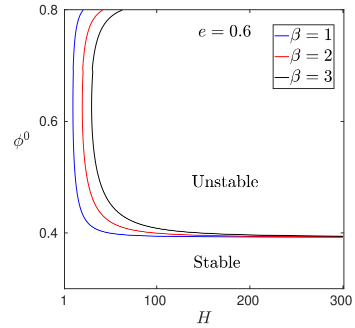

Figure 1. Neutral stability curves in (H, ϕ^0)-plane for $e = 0.6$.

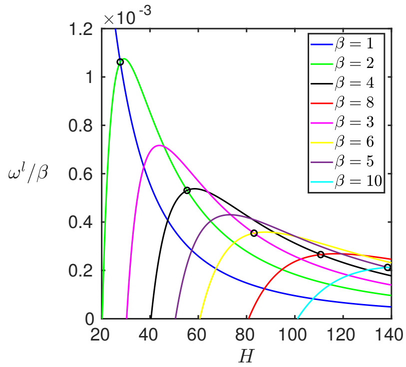

Figure 2. Variation of ω^l/β with H for various mode number β with $\phi^0 = 0.6$ and $e = 0.6$. The circled interaction points represent the locations of 1:2 resonances.

H	β_1	β_2	ω_1^l	ω_2^l
27.6905	1	2	1.062E-03	2.12402E-03
55.3803	2	4	1.062E-03	2.12401E-03
83.0703	3	6	1.062E-03	2.12401E-03
110.7604	4	8	1.062E-03	2.12401E-03
138.4505	5	10	1.062E-03	2.12401E-03

Table 1. Couette gap, mode number and frequency for 1:2 resonance. Here $\omega_1^l : \omega_2^l = \beta_1 : \beta_2$ =1:2; see figure 2.

at $H = H_1$, then $(2\beta_1, \omega_1)$ and $(2\beta_2, \omega_2)$ will also satisfy $\omega_1 : \omega_2 = 2\beta_1 : 2\beta_2$ =1:2 at $H = 2H_1$, and so on. This suggests that if 1:2 resonance exists between modes 1 and 2 at $H = H_1$, then a similar resonance arises between modes 2 and 4 at $H = 2H_1$, between modes 3 and 6 at $H = 3H_1$, and so on. The repeating pattern of these resonances is visually evident in the intersections of different colored lines in Figure 2 and the parameter values listed in Table 1. This systematic resonance cascade suggests that increasing the scaled plate separation H does not introduce random mode interactions but instead leads to a structured sequence of resonances between higher order harmonics. In the next section, we derive the general form of the amplitude equations under 1:2 resonances, and then analyze the particular case of mode 1 interacting with mode 2.

5 Coupled amplitude equations

The coupled amplitude equations have been thoroughly derived in Ref. [7]. For the sake of completeness, here we briefly outline the derivation. The nonlinear disturbance equations can be expressed as, $(\partial/\partial t - \mathcal{L})\, X = \mathcal{N}_2 + \mathcal{N}_3$, where $\mathcal{L}$ is the linear operator, $\mathcal{N}_2$ and $\mathcal{N}_3$ are the quadratic and cubic nonlinear terms, respectively. The analytical solution of a homogeneous problem is known from LSA described in Sec.3. We write disturbance X as $X = \Phi + \Psi$, where Φ corresponds to the resonating modes, and Ψ to all other modes: $X(y,t) = \Phi(y,t) + \Psi(y,t)$, with $\Phi(y,t) = \mathcal{A}_1(t)E_1X^{[1;1]}(y) + \mathcal{A}_2(t)E_2Y^{[1;1]}(y)$ + c.c, where $X^{[1;1]}(y)$ and $Y^{[1;1]}(y)$ are the eigenfunctions corresponding to the eigenvalue with the maximum real part of the linearized problem at mode numbers β_1 and β_2, respectively. Thus, $L_{\beta_1}X^{[1;1]} = \omega_1^l X^{[1;1]}$ and $L_{\beta_2}Y^{[1;1]} = \omega_2^l Y^{[1;1]}$, where $L_{\beta_1} = L_1(\beta = \beta_1)$ and $L_{\beta_2} = L_1(\beta = \beta_2)$. $E_1 = e^{\omega_1^l t}$ and $E_2 = e^{\omega_2^l t}$ are the exponential contributions of the resonating modes. Writing solution of nonlinear disturbance equations as

$$X(y,t) = \sum_{k=-\infty}^{\infty}\left[X^{(k)}(y,t)E_1^k + Y^{(k)}(y,t)E_2^k\right] + \left[\sum_{i,j\geq 0, i=j\neq 0}^{\infty} Z^{(ij)}(y,t)E_1^iE_2^j + c.c.\right], \quad (3)$$

where $X^{(k)}$ represents the amplitudes of the modes generated by the interactions of the first resonating mode (the mode with growth rate ω_1^l, mode number β_1, amplitude $\mathcal{A}_1$) with its different harmonics. Similarly, $Y^{(k)}$ represents the amplitudes of the modes generated by the interactions of the second resonating mode (the mode with growth rate ω_2^l, mode number β_2, amplitude $\mathcal{A}_2$) with its different harmonics. $Z^{(ij)}$ denotes amplitudes of the modes generated by the interactions of $X^{(k)}$ and $Y^{(k)}$. Substituting (3) into the nonlinear disturbance equation and collecting the coefficients of the same power of E_1, E_2 we obtain :

$$\begin{aligned}\left(\frac{d}{dt} - \omega_1^l\right)\mathcal{A}_1X^{[1;1]} &= \mathcal{N}_2(X^{(1)}, X^{(0)}) + \mathcal{N}_2(X^{(0)}, X^{(1)}) + \mathcal{N}_2(\tilde{X}^{(1)}, X^{(2)}) \\ &+ \mathcal{N}_2(X^{(2)}, \tilde{X}^{(1)}) + \mathcal{N}_2(X^{(1)}, Y^{(0)}) + \mathcal{N}_2(Y^{(0)}, X^{(1)}) \\ &+ \mathcal{N}_2(Z^{(11)}, \tilde{Y}^{(1)}) + \mathcal{N}_2(\tilde{Y}^{(1)}, Z^{(11)}) + \mathcal{N}_2(Y^{(1)}, Z^{(1-1)}) \\ &+ \mathcal{N}_2(Z^{(1-1)}, Y^{(1)}) + \underline{\mathcal{N}_2(\tilde{X}^{(1)}, Y^{(1)}) + \mathcal{N}_2(Y^{(1)}, \tilde{X}^{(1)})} + \ldots, \quad (4)\end{aligned}$$

$$\begin{aligned}\left(\frac{d}{dt} - \omega_2^l\right)\mathcal{A}_2Y^{[1;1]} &= \mathcal{N}_2(Y^{(1)}, Y^{(0)}) + \mathcal{N}_2(Y^{(0)}, Y^{(1)}) + \mathcal{N}_2(\tilde{Y}^{(1)}, Y^{(2)}) \\ &+ \mathcal{N}_2(Y^{(2)}, \tilde{Y}^{(1)}) + \mathcal{N}_2(Y^{(1)}, X^{(0)}) + \mathcal{N}_2(X^{(0)}, Y^{(1)}) \\ &+ \mathcal{N}_2(Z^{(11)}, \tilde{X}^{(1)}) + \mathcal{N}_2(\tilde{X}^{(1)}, Z^{(11)}) + \mathcal{N}_2(X^{(1)}, Z^{(-11)}) \\ &+ \mathcal{N}_2(Z^{(-11)}, X^{(1)}) + \underline{\mathcal{N}_2(X^{(1)}, X^{(1)})} + \ldots. \quad (5)\end{aligned}$$

The underlined terms arise due to the 1:2 resonance interaction of $X^{[1;1]}$ and $Y^{[1;1]}$. To reduce the dimensionality, we use the center manifold reduction method [5], which helps to make the system four dimensional consists of $\mathcal{A}_1$, $\mathcal{A}_2$, $\bar{\mathcal{A}}_1$ and $\bar{\mathcal{A}}_2$. The center manifold theorem states that the dynamics in the neighborhood of the critical conditions is governed by a low dimensional system, which can be obtained by projecting the infinite dimensional non-critical modes onto critical modes, i.e. $\Psi = \Psi(\Phi)$ [for more details see [5] and the references therein]. Thus we can approximate noncritical modes as the function of critical modes using the Taylor series expansion as follows:

$$\begin{aligned}X^{(0)}(y,t) &= |\mathcal{A}_1|^2\left(X^{[0;2]} + |\mathcal{A}_1|^2 X^{[0;4]} + \cdots\right)\\ X^{(1)}(y,t) &= \mathcal{A}_1\left(X^{[1;1]} + |\mathcal{A}_1|^2 X^{[1;3]} + \cdots\right)\\ X^{(2)}(y,t) &= \mathcal{A}_1^2\left(X^{[2;2]} + |\mathcal{A}_1|^2 X^{[2;4]} + \cdots\right)\\ Y^{(0)}(y,t) &= |\mathcal{A}_2|^2\left(Y^{[0;2]} + |\mathcal{A}_2(t)|^2 Y^{[0;4]} + \cdots\right)\\ Y^{(1)}(y,t) &= \mathcal{A}_2\left(Y^{[1;1]} + |\mathcal{A}_2|^2 Y^{[1;3]} + \cdots\right)\\ Y^{(2)}(y,t) &= \mathcal{A}_2^2\left(Y^{[2;2]} + |\mathcal{A}_2|^2 Y^{[2;4]} + \cdots\right) \quad (6)\\ Z^{(1-1)}(y,t) &= \mathcal{A}_1\tilde{\mathcal{A}}_2\left(Z^{[0;2]}_{1-1} + |\mathcal{A}_1|^2 Z^{[0;4]}_{1-1X} + |\mathcal{A}_2|^2 Z^{[0;4]}_{1-1Y} + \cdots\right)\\ Z^{(-11)}(y,t) &= \tilde{\mathcal{A}}_1\mathcal{A}_2\left(Z^{[0;2]}_{-11} + |\mathcal{A}_1|^2 Z^{[0;4]}_{-11X} + |\mathcal{A}_2|^2 Z^{[0;4]}_{-11Y} + \ldots\right)\\ Z^{(10)}(y,t) &= \mathcal{A}_1\left(|\mathcal{A}_2|^2 Z^{[1;3]}_{10Y} + |\mathcal{A}_2|^4 Z^{[1;5]}_{10Y} + \ldots\right)\\ Z^{(01)}(y,t) &= \mathcal{A}_2\left(|\mathcal{A}_1|^2 Z^{[1;3]}_{01X} + |\mathcal{A}_1|^4 Z^{[1;5]}_{01X} + \ldots\right)\\ Z^{(11)}(y,t) &= \mathcal{A}_1\mathcal{A}_2\left(Z^{[2;2]}_{11} + |\mathcal{A}_1|^2 Z^{[2;4]}_{11X} + |\mathcal{A}_2|^2 Z^{[2;4]}_{11Y} + \ldots\right)\end{aligned}$$

where subscripts X and Y denote the mean flow contributions from $X^{(k)}$ and $Y^{(k)}$ modes, respectively. Replacing the expressions $X^{(k)}$, $Y^{(k)}$ and $Z^{(ij)}$ in the nonlinear terms of (4) and (5) by the above ansatz (6) and simplifying the obtained expression, we get the coupled amplitude equations truncated at the cubic order,

$$\begin{aligned}\left(\frac{d}{dt} - \omega_1^l\right)\mathcal{A}_1X^{[1;1]} &= G^{11}_{13}\mathcal{A}_1|\mathcal{A}_1|^2 + G^{12}_{13}\mathcal{A}_1|\mathcal{A}_2|^2 + G^{13}_{13}\tilde{\mathcal{A}}_1\mathcal{A}_2,\\ \left(\frac{d}{dt} - \omega_2^l\right)\mathcal{A}_2Y^{[1;1]} &= G^{21}_{13}\mathcal{A}_2|\mathcal{A}_1|^2 + G^{22}_{13}\mathcal{A}_2|\mathcal{A}_2|^2 + G^{23}_{13}\mathcal{A}_1^2,\end{aligned} \quad (7)$$

where G^{11}_{13}, G^{12}_{13}, G^{21}_{13}, G^{22}_{13}, G^{13}_{13} and G^{23}_{13} are functions of the higher order harmonics of the resonating fundamental modes, and harmonics satisfy the following equations

$$\begin{aligned}(2\omega_1^l - L)X^{[0;2]} &= N_2(X^{[1;1]}, \tilde{X}^{[1;1]}) + N_2(\tilde{X}^{[1;1]}, X^{[1;1]}),\\ (2\omega_1^l - L)X^{[2;2]} &= N_2(X^{[1;1]}, X^{[1;1]}),\\ (2\omega_2^l - L)Y^{[0;2]} &= N_2(Y^{[1;1]}, \tilde{Y}^{[1;1]}) + N_2(\tilde{Y}^{[1;1]}, Y^{[1;1]}),\\ (2\omega_2^l - L)Y^{[2;2]} &= N_2(Y^{[1;1]}, Y^{[1;1]}),\\ ((\omega_1^l + \omega_2^l) - L)Z^{[0;2]}_{-11} &= N_2(Y^{[1;1]}, \tilde{X}^{[1;1]}) + N_2(\tilde{X}^{[1;1]}, Y^{[1;1]}),\\ ((\omega_1^l + \omega_2^l) - L)Z^{[0;2]}_{1-1} &= N_2(\tilde{Y}^{[1;1]}, X^{[1;1]}) + N_2(X^{[1;1]}, \tilde{Y}^{[1;1]}).\end{aligned}$$

Here, $L = \mathcal{L}(\partial/\partial y \to d/dy)$. Taking the inner product of (7) with adjoint eigenfunctions $X^{\dagger}$ and $Y^{\dagger}$, respectively, and using bi-orthogonality, we obtain the coupled amplitude equations:

$$\frac{d\mathcal{A}_1}{dt} = \omega_1^l \mathcal{A}_1 + \lambda_{11}\mathcal{A}_1|\mathcal{A}_1|^2 + \lambda_{12}\mathcal{A}_1|\mathcal{A}_2|^2 + \lambda_{13}\tilde{\mathcal{A}}_1\mathcal{A}_2,$$
$$\frac{d\mathcal{A}_2}{dt} = \omega_2^l \mathcal{A}_2 + \lambda_{21}\mathcal{A}_2|\mathcal{A}_1|^2 + \lambda_{22}\mathcal{A}_2|\mathcal{A}_2|^2 + \lambda_{23}\mathcal{A}_1^2, \quad (8)$$

where λ_{ij}'s are defined in terms of inner products [7]. Substituting $\mathcal{A}_1 = \rho_1 e^{\theta_1}$ and $\mathcal{A}_2 = \rho_2 e^{\theta_2}$ and separating the real and imaginary parts, we get equations for ρ_1, ρ_2 and $\Theta = \theta_2 - 2\theta_1$:

$$\frac{d\rho_1}{dt} = \omega_1^l \rho_1 + \lambda_{11}\rho_1^3 + \lambda_{12}\rho_1\rho_2^2 + \lambda_{13}\rho_1\rho_2\cos\Theta,$$
$$\frac{d\rho_2}{dt} = \omega_2^l \rho_2 + \lambda_{21}\rho_1^2\rho_2 + \lambda_{22}\rho_2^3 + \lambda_{23}\rho_1^2\cos\Theta, \quad (9)$$
$$\frac{d\Theta}{dt} = -\left(\lambda_{23}\frac{\rho_1^2}{\rho_2} + 2\lambda_{13}\rho_2\right)\sin\Theta.$$

The equilibrium solutions are found to be non-traveling (standing) waves, given by:

$$\rho_{1\pm} = -\frac{1}{\lambda_{11}}\left(\omega_1^l + \lambda_{12}\rho_{2\pm}^2 \pm \lambda_{13}\rho_{2\pm}\right), \quad \text{where } \rho_{2\pm} \text{ satisfy}$$
$$(\lambda_{21}\lambda_{12} - \lambda_{11}\lambda_{22})\rho_{2\pm}^3 \pm (\lambda_{21}\lambda_{13} + \lambda_{23}\lambda_{12})\rho_{2\pm}^2 + \left(\omega_1^l\lambda_{21} - \omega_2^l\lambda_{11} + \lambda_{23}\lambda_{13}\right)\rho_{2\pm} \pm \lambda_{23}\omega_1^l = 0$$
$$\Theta_+ = 2n\pi, \quad \text{and} \quad \Theta_- = (2n+1)\pi, \quad \text{for} \quad n = 0, 1, 2, 3...$$

The linear stability of the coupled amplitude equations (9) around these equilibrium points is determined by evaluating the eigenvalues of the corresponding Jacobian matrix at the resonating parameters. This analysis reveals the presence of one unstable equilibrium and one saddle-type equilibrium for the parameters mentioned in the first row of Table 1. The phase space dynamics in (ρ_1, ρ_2) plane around these equilibrium points is shown in figure 3, where the red and green circles represent the unstable node and saddle-type equilibrium points, respectively, highlighting the instability of the resonating modes.

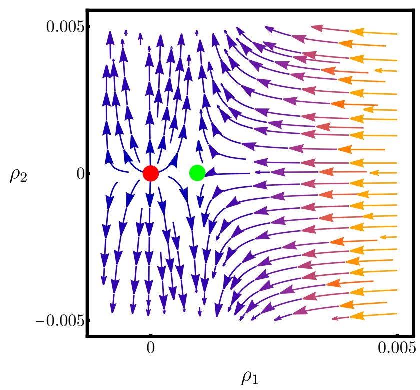

Figure 3. Streamlines in (ρ_1, ρ_2) plane, where ρ_1 and ρ_2 are absolute amplitudes of resonating modes $\beta_1 = 1$ and $\beta_2 = 2$, respectively, at $H = 27.69$. Other parameters are the same as figure 2. The red and green markers highlight unstable node and saddle equilibrium points, respectively.

6 Conclusion

A semi-analytical framework has been developed to determine parameters for 1:2 resonant interaction in uniform GCF. The coupled amplitude equations (8) describing the amplitudes of shearbanding modes under 1:2 resonances have been derived in detail. Additionally, their equilibrium solutions have been analytically obtained, especially of the mixed-mode type. Furthermore, the method outlined here can also be extended to derive amplitude evolution equations for other higher harmonic resonances, for instance, 1:3 and 1:2:3 resonances. The coefficients of the amplitude equations are used to predict the dynamical behavior of resonating shearbanding modes. For the chosen parameters, the present analysis has revealed the existence of two unstable equilibrium solutions. These findings advance the theoretical understanding of harmonic resonant interactions in uniform GCF. This model can be extended to a more realistic scenario by considering arbitrary elasticity, particle size, and volume fraction in a three-dimensional configuration. Furthermore, the comparison of the present work with the simulations and experiments will be carried out in the future.

Acknowledgment

P.S. acknowledges Prof. Meheboob Alam for initial discussions on the resonance phenomenon in granular flows.

References

[1] I. Goldhirsch, Rapid granular flows, Annu. Rev. Fluid Mech. **35**, 267 (2003). 10.1146/annurev.fluid.35.101101.161114
[2] M.R.E. Proctor, C.A. Jones, The interaction of two spatially resonant patterns in thermal convection. part 1. exact 1:2 resonance, J. Fluid Mech. **188**, 301 (1988). 10.1017/S0022112088000746
[3] P. Shukla, L. Biswas, V.K. Gupta, Shear-banding instability in arbitrarily inelastic granular shear flows, Phys. Rev. E **100**, 032903 (2019). 10.1103/PhysRevE.100.032903
[4] M. Alam, P.R. Nott, Stability of plane couette flow of a granular material, J. Fluid Mech. **377**, 99 (1998). 10.1017/S002211209800295X
[5] P. Shukla, M. Alam, Landau-type order parameter equation for shearbanding in granular Couette flow, Phys. Rev. Lett. **103**, 068001 (2009). 10.1103/PhysRevLett.103.068001
[6] P. Shukla, M. Alam, Nonlinear stability and patterns in granular plane couette flow: Hopf and pitchfork bifurcations, and evidence for resonance, Journal of Fluid Mechanics **672**, 147–195 (2011). 10.1017/S002211201000594X
[7] P. Shukla, Ph.D. thesis, JNCASR (2011)