

Validation of the most probable disordered packings by the experiments on the loose and close states

Calixtro Yanqui ^{1,*}

¹Civil Engineering Department, National University of Saint Augustine of Arequipa, Peru

Abstract. The theoretical and experimental determination of the loose and close states of grain assemblages is of great importance in the behaviour of granular materials. Obviously, the most appropriate basic model to achieve this objective is the disordered packing of spheres. In this work, it is analysed the conditions under which the theory of most probable disordered packings is compatible with the careful experiments carried out by various authors. For loose packings, the theoretical data are compared with the experiments on pouring balls into a container, and liquefaction and sedimentation in water, and very dense liquid as well. For close packing, the most probable volume fraction is validated with the experimental data obtained by compression and kneading in an elastic balloon, vertical tapping, vertical compression and horizontal shear vibration. For critical packing state, granular impact and pulling force method are considered. The fundamental conclusion of the study is that the fit of the theoretical and experimental values is very good.

1 Introduction

The most probable packing is a theoretical model based on randomizing the angles of a regular packing of spheres. By this operation, the expected volume fraction of a random packing of spheres is obtained. The model predicts several limiting values, depending on the packing symmetry, the type of strain, and the probability condition [1]. This paper addresses the validation of these values in light of experiments carried on by various authors around the world, which, fortunately, cover all possibilities. First, the details and results of the experiments are described; then, these are compared with the theoretical data, and finally, a simple interpretation is given.

2 Loose state

2.1 By pouring into a container

Experiments. To obtain the loose state of a random packing, the pioneering authors carried out experiments, that consisted of pouring through a funnel, in the air,

balls of uniform size and composition into a container of sufficiently large volume [2][3]. In this way, specific densities of the order of 0.59 to 0.61 were obtained. Scott [2] used steel balls and obtained a specific density, corrected for the effects of the extremes, of 0.60. Owe Berg et al. [4] conducted experiments with an aggregate of steel balls poured into a cylinder, and obtained specific densities between 0.586 and 0.592 for the loose state of the packing, depending on the spheres drop height. In order to get a vertically uniform distribution, it has been proposed to pour the spheres into an inner container resting at the base of the outer container. The spheres used by Vanel and Clement [5] were composed of glass, rough quartz, and steel. The inner container was then slowly pulled, allowing the mass of spheres to flow gently. The experimental specific density obtained was 0.583. To achieve a better effect, it has been proposed to provide a sieve at the base of the inner cylinder. A variant of this method is to place a rod in the axis of the cylindrical container before pouring the spheres, and then pull it slowly. Aste et al. [6] used acrylic spheres and obtained specific densities of 0.586 and 0.596, depending on the number of spheres

* Corresponding author: cyanqui@unsa.edu.pe

Theoretical interpretation. The uniform pouring of the spheres yields better results than the funnel procedure, and is comparable to the specific density of 0.586, that corresponds to a completely random three-dimensional packing, whose joint probability density function is bivariate.

2.2 By liquefaction and sedimentation in water

Experiments. Schrotër et al. [7] applied controlled pulses of vertical deionized water flow to an assemblage of glass spheres composed of sodium carbonate, with an initial volume fraction of 0.581, by pouring the spheres. The container used was a tube of a square section. The flow pulses suspended the spheres getting a viscous bed of stationary height. The bed was then allowed to settle. Applying an appropriate fitting function, the authors found a specific density of 0.573. Experiments with beads in water done by Aste et al. [6] yielded specific density values ranging from 0.56 to 0.60, depending on the flow rate. Using a similar procedure, Métayer et al. [8] obtained a specific density of 0.57.

Theoretical interpretation. The upward vertical flow of water lifts the spheres and tends to disjoin the original, completely random packing. The new packing formed by sedimentation from a very low height is configured by compression strain. Consequently, the arrangement corresponds to a conditioned, three-dimensional compressional random packing, for which the specific density is 0.574.

2.3 By sedimentation in a very dense liquid

Experiments. Observing that the volume fraction of the packing found by pouring method, in air, depended on the specific gravity of the balls, some authors suggested methods to avoid or reduce the effect of gravity. For example, Onoda and Liniger [9] allowed the spheres to settle gently in a liquid of specific gravity close to that of the spheres. The experiment was carried out in a 500 ml graduated cylinder with 0.25 mm diameter glass balls, finding a volume fraction of 0.555. Farrel et al. [10] used an hourglass-shaped device made up of 60° opening cones and a base diameter of 24 spheres, connected by a cylindrical neck with a diameter of 4.2 spheres, thus reducing the arching effect of the assemblage and allowing a small number of balls to pass through per unit of time. The procedure consisted of filling the base cone, then inverting the device filled with a selected dense liquid, and letting the balls to drop, in similar way to the funnel method. The specific gravities found were 0.551 for the steel balls and 0.540 for the acrylic balls. Jerkins et al. [11] optimized the previous procedure by settling 0.25 mm diameter glass balls, after which they applied vertical flow pulses to the assemblage. Adding sodium polytungstate to the water achieved liquid densities very close to that of the balls. The container was a vertical polycarbonate tube with an internal diameter of 12.8 mm and a length of 230 mm. A porous bronze disc was placed at the base of this tube, and the flow pulses were caused by a pump, maintaining a constant flow rate for 2.0 minutes, until reaching a

stationary height. They then allowed the balls to settle to form a loose assemblage, noting that by increasing the settling time, the specific density could be reduced to 0.550.

Theoretical interpretation. The packing generated by the sedimentation process in a very dense liquid is random in the horizontal plane and determined in the vertical plane. As occurs with sedimentation in water, the vertical flow caused by falling spheres reaches its maximum efficiency when the drag force tends to form columns. This condition corresponds to a conditional random packing, polar or azimuthal, which is strained by the drag exerted on the spheres by the vertical flow of the viscous fluid. For this condition, the specific density 0.548.

3 Close state

3.1 By compression and kneading

Experiment. Bernal and Mason [12] compressed and kneaded an assemblage of metal balls in a balloon with an elastic membrane stretched by the filling. The volume fraction obtained after a sufficiently long time was 0.637.

Theoretical Interpretation. As a first approach, kneading involves the rolling of the sample, and therefore, the deformation mechanism is essentially two-dimensional. Therefore, the resulting element can be described as a marginal packing, random in the direction of one coordinate axis and determinate in the other. For the close state, the specific density is 0.633.

3.2 By discrete vertical taps

Experiments. Once the loose assemblage of steel balls was pouring to get a specific density between 0.586 and 0.592, Owe Berg et al. [4] placed the cylinder containing them on a piston, resting on a rotating cam with four symmetrical notches, connected to the shaft of a horizontal motor. The notches lifted the piston and produced the impacts. After 400 shocks, the motor was turned off. Repeating this operation gave a specific density of 0.615. Vanel and Clément [5] increased the volume fraction by means of vibrations produced by a series of taps on the walls of the container, reaching a maximum value of 0.62. Knight et al. [13] measured a specific density of 0.61 to 0.62 for an assemblage of glass beads in a long, thin tube, subjected to moderate intensity taps, lasting several minutes. Philippe and Bideau [14] used glass beads to fill a glass cylinder. The cylinder was shaken at regular intervals by an electromagnetic exciter that imposed vertical impacts in a harmonic manner. To standardize the results, all experiments started with a specific density of 0.583 and were subjected to different accelerations. The results ranged between 0.60 and 0.625, reaching the peak when the applied acceleration was 1.2 times the acceleration of gravity. The aim of Tariot et al. [15] was the densification of an assemblage of spheres by fluidization. Once a specific density of 0.605 for the loose state was obtained by fluidization and

sedimentation, the liquid mixture of water and glycerol was continuously pumped at a speed lower than a certain critical speed that depended on Stokes' law. For continuous flow, the assemblage reached a specific density of 0.62, after approximately 10 seconds, and 0.625, after 3.0 days.

Theoretical interpretation. First, these experiments validate the completely random nature of the loose state in the free condition, above described, with a specific volume of 0.586. Second, discrete vertical shaking changes the initial, completely random packing to a conditionally compressive three-dimensional packing; random in the horizontal plane and determined in the vertical plane, whose final volume fraction reaches the value of 0.62.

3.3 By harmonic vertical vibration

Experiments. Nowak et al. [16] filled a tube with beads made of sodium carbonate glass. Then, the tube was subjected to discrete vertical shaking consisting of a complete cycle of a harmonic wave. The loose state of the assemblage was achieved by flowing high pressure dry nitrogen from the base to the top of the tube, establishing a volume fraction of 0.580. In the densification stage, the acceleration amplitude was slowly and successively increased, maintaining 105 shakings for all experiments. Once an amplitude of 6.7 times the acceleration of gravity was reached, the impact acceleration was systematically decreased. The hysteric curve shows that the process is irreversible up to an acceleration of 3.0 times the acceleration of gravity, associated to a peak specific density of 0.637 on the loading curve, and a final specific density of 0.647, on the discharge curve. Zhang and Rosato [17] used acrylic spheres to fill a cylinder formed by overlapping rings fixed to a B&K shaker. The spheres were slowly poured into the container, exhibiting a 0.604 specific density. The curves deduced from the shaking densification showed a peak specific density of 0.636 in a range of 5 to 6 times the acceleration of gravity. The authors found that, for a frequency of 40 Hz and a certain diameter of the spheres, the specific density reached the value of 0.655. An et al. [18] studied the effect of various operating conditions on the densification of an assemblage subjected to vertical vibration, such as vibration amplitude and frequency, test time, feeding method, container size and material properties. By extrapolation of data from different container sizes, the authors found that the maximum volume fraction of the glass sphere assemblage was 0.634 for the total feed before starting the vibration, and 0.653 for the controlled batch feed during vibration. Tariot et al. [15] observed that applying flow bursts could also densify a fluidized assemblage of spheres. Using a pump in the same equipment described above, they managed to develop an intermittent upward flow, having found that two steps were required to reach irreversibility: applying hard taps and, then, decreasing their intensity. For the first step, the specific volume was 0.63, and for the second, 0.655.

Theoretical interpretation. In all the experiments described, harmonic densification with accelerations

between 5 and 7 times the acceleration of gravity obeys the same statistical principles that govern tap densification, both for the initial and compacted states. Thus, the specific density values are 0.586 for the pouring and 0.633 for the random packing densified by vibration. The volume fraction is generally increased by unloading, sphere diameter, amplitude, frequency, and duration of vibration; up to 0.655 for a conditional polar three-dimensional packing, which is formed from a joint random packing.

3.4 By unidirectional shear vibration

Experiments. Pouliquen et al. [19] subjected a glass sphere assemblage container to horizontal vibration while the spheres were poured at a constant flow rate. The authors performed tests with different accelerations and found that, at low accelerations, the specific density was 0.625, regardless of the pouring rate, and that, at higher accelerations, the specific density increased to 0.670 and was highly dependent on the filling rate and the acceleration itself. Li et al. [20] used equipment that allowed the container to vibrate in three directions. However, in the reported study, they used the same frequency and amplitude in all three directions, so the phase difference was zero and the motion was linear. The PMMA container was filled by slowly pouring glass spheres. This operation was done in two ways: by total feeding before vibration, and by feeding in weighed batches during vibration. In the first mode, the abscissa diagrams, describing the different parameters involved, lead to the conclusion that the peak specific density was 0.670, on average. In the second mode, the specific gravity of samples formed by a large number of layers per batch tends to 0.70, on average. Nicolas et al. [21] modified the container into a prismatic box with a base and side walls fixed to the table, while the other two walls could rotate around linear pins fixed to the table. This arrangement allowed for the imposition of a horizontal rectangular plate on the top and the application of a horizontal vibratory motion to the base. The movable side walls of the container were kept in contact with the horizontal top plate by two cables. The material used consisted of spherical glass beads, and the specific density achieved was 0.693, either through continuous or discontinuous vibration in stages. The authors found that the assemblage of spheres included crystalline arrangements in the mass.

Theoretical interpretation. According to the experiments, densification by horizontal vibration depends on several factors. If the deposit is a rigid box, the specific gravity depends on the magnitude of the acceleration. When the acceleration is low, the acceleration of gravity prevails and the specific density is 0.62, demonstrating that vibration randomizes the packing in the horizontal plane. A greater acceleration prioritizes the role of horizontal shear deformation, reaching a specific density of 0.672, which corresponds to a conditional polar random packing. Pouring in weighted batches into a rigid vibrating box or using a box with two opposite reciprocating faces leads to looser crystallization of the packing by shear deformation. For this condition, the theoretical specific density is 0.698.

4 Critical state

Experiments. Umbanhowar and Goldman [22] determined the critical state volume fraction by performing impact tests by propelling a steel ball on a bed made of glass spheres contained in a 30x24x24 cm³ box. They prepared samples with different volume fractions and measured the volumetric change after the impact, observing the sign changed for a volume fraction of 0.591. Metayer et al. [8] found a value of 0.595 for the specific density that separates dilatant packings from contractive ones, measuring the pulling force of a plate previously placed in a vertical glass tube.

Theoretical interpretation. These experiments demonstrates that the critical packing state corresponds to a joint random three-dimensional packing, whose probability density function is 0.586.

5 Conclusions

The experimental data on the specific volume for free-dropping spheres into a container are consistent with those obtained assuming a joint or total probability density function. Results by moderately constrained experiments, such as sedimentation in a dense liquid or low-acceleration vibration, fit well with the specific volumes obtained for a marginal or conditional probability. Data from highly constrained experiments are consistent with those for anisotropic regular packings. Figure 1 shows this good agreement.

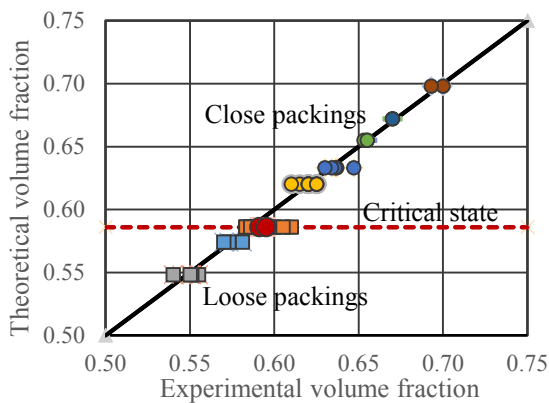


Fig. 1. Comparison of the most probable volume fractions with the respective experimental values.

References

1. C. Yanqui, Specific volume of the most probable disordered sphere packings. *Powders and Grains*, 2025, (To be published).
2. G.D. Scott. *Nature*. **188**, 908-909, (1960)
3. R. Rutgers, Packing of spheres, *Nature*, (1962)
4. T.G. Owe Berg, R.L. McDonald, R.J. Trainor, Jr. The packing of spheres, *Powder Technol.*, **3**, 183-188 (1969)
5. L. Vanel, E. Clément, Pressure screening and fluctuations at the bottom of a granular column, *Eur. Phys. J. B* **11**, 525-523 (1999)

6. T. Aste, M. Saadatfar, A. Sakellariou, T.J. Senden, Investigating the geometrical structure of disordered sphere packings, *Physica A*, **339**, 16-23, (2004)
7. M. Schröter, D.I. Goldman, H.L. Swinney, Stationary state volume fluctuations y a granular medium, *Phys. Rev. E* **71**, 030301(R), (2005)
8. J.F. Métayer, D.J. Suntrup III, C. Radin, H.L. Swinney, M. Schröter, Shearing of frictional sphere packings. *EPL* **93** (6), 64003 (2011)
9. G.Y. Onoda, E.G. Liniger, Random loose packings of uniform spheres and the dilatancy onset, *Phys. Rev. Lett.* **64**, 2727-2730, (1990)
10. G.R. Farrel, K.M. Martini, N. Menon, Loose packings of frictional spheres, *Soft Matter*, **6**, 2925-2930, (2010)
11. M. Jerkins, M. Schroter, H.L. Swinney, Onset of mechanical stability in random packings of frictional spheres, *Phys. Rev. Lett.* **101**, 018301 (2008)
12. J.D. Bernal, J. Mason, Packing of spheres: coordination of randomly packed spheres. *Nature* **188**, 910-911 (1960)
13. J.B. Knight, C.G. Fandrich, C. Ning Lau, H.M. Jaeger, S.R. Nagel, Density relaxation in a vibrated granular material. *Phys. Rev. E* **51**, 3957 (1995)
14. P. Philippe, D. Bideau, Compaction dynamics of a granular medium under vertical tapping, *EPL* **60** 677 2002)
15. A. Tariot, G. Gauthier, P. Gondret, Granular compaction by fluidization, *EPJ Web of Conf.* **140**, 10003 (2017)
16. E. R. Nowak, J.B. Knight, E. Ben-Naim, H.M. Jaeger, S.R. Nagel, Density fluctuations in vibrated granular materials, *Phys. Rev. E* **57** (2) (1998)
17. N. Zhang, A.D. Rosato, Experiments and simulations on vibration induced densification of bulk solids, *KONA Powder* **24**, 93-103 (2006)
18. X.Z. An, C.X. Li, R.Y. Yang, R.P. Zou, A.B. Yu, Experimental study on the packing of mono-sized spheres subjected to one-dimensional vibration, *Powder Technology* **196**, 50-55 (2009)
19. O. Pouliquen, M. Nicolas, P.D. Weidman, Crystallization of non-brownian spheres under horizontal shaking, *Phys. Rev. Lett.* **79**. 3640-3643, (1997)
20. C.X. Li, X.Z. An, R.Y. Yang, R.P. Zou, A.B. Yu, Experimental study on the packing of uniform spheres under three-dimensional vibration. *Powder Technology* **208**, 617-622 (2011)
21. M. Nicolas, P. Duru, O. Pouliquen, Compaction of a granular material under cyclic shear. *European Phys. J. E*, **3**, 309-314, (2000)
22. P. Umbanhowar, D.I. Goldman, Granular impact and the critical packing state. *Phys. Rev. E* **82**, 010301(R) (2010)