

# Bins discharge limited by elastic gates

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**Abstract.** For circular orifices of diameter  $D$ , located at the bottom of silos, the gravity-driven mass flow rate,  $m'$ , of dry, non-cohesive granular material follows the Hagen formula given by  $m' \sim D^{5/2}$ , which, to our knowledge, is independent of the granular pressure near the exit orifice. In this work, we report experiments in nearly two-dimensional bins where narrow rectangular elastic plates of small thickness were attached of known thickness  $h$ , length,  $l$ , and Young's modulus,  $E$ , that can be deflected a distance  $\zeta$ , by the granular pressure acting on the plate  $p_l$ . For this case, it was experimentally found that for this case, the mass flow rate is dependent upon all these quantities, but mainly on  $\zeta$ , as a power law of the form  $m' \sim \zeta^{5/2}$ , whilst the pressure seems to behave as  $p_l \sim 1/l$ . Our analysis reveals the link between the flow rate and the local pressure, a feature that might find practical applications.

## 1 Introduction

Non-cohesive granular matter is complex, for instance, the pressure associated with the static storage of grains in a silo deviates substantially from the hydrostatic pressure, exerted by a liquid, which varies linearly with depth. In a silo, the pressure at the bottom saturates since a large portion of the weight is supported by the walls. This is known as the Janssen effect [1]. Moreover, previous to the work of Janssen, Hagen [2] found that for the gravity driven flow of grains the mass flow rate,  $m'$ , through a circular exit hole of diameter  $D$ , follows the relationship  $m' \sim D^{5/2}$ , which is independent of the filling level in the silo.

Since the initial works of Janssen [1] and Hagen [2], a general connection between the mass flow of grains and pressure has not been established, as occurs in the Bernoulli equation in ideal fluids [3]. Even today, real world cases are still being explored [4,5].

In this work, it is analyzed a configuration where becomes necessary to establish a relationship between the granular flow and the pressure exerted by the grains,  $p_l$ , on an elastic plate which is placed at the silo orifice. This plate of length  $l$ , thickness  $h$  and Young's modulus  $E$ . The pressure causes a deflection of the plate  $\zeta$  [6], which generates a gap through which a granular flow  $m'$  develops. The relationship between  $m'$ ,  $\zeta$  and  $p_l$  will be studied as the main objective of this paper.

To reach our goal, in next Section we revisit the Janssen and Hagen formulas for the pressure and mass flow rate, respectively, in a rectangular near two-dimensional bin. Later on, in Section 3, we will also revisit the problem of the large deflection of an elastic plate in order to establish a connection between the local

pressure, due to the granular material, and the mass flow rate. Experiments will be presented to shed light on fundamental concepts. Also, our theoretical correlations and data analysis will be based on order of magnitude estimations, to gain physical insights of the phenomena. Finally, in Section 4 we will give the main conclusions of this work.

## 2 Pressure and mass flow in a silo

### 2.1 Janssen pressure

The central observation is that in a silo the vertical pressure, measured with gauges at the bottom, is generally much smaller than the hydrostatic pressure  $\rho g H$ , which would be present in a liquid ( $\rho$ : bulk density,  $g$ : gravity acceleration,  $H$ : column height. See Fig. 1). A first modelization for this phenomenon was given long ago by Janssen who carried experiments in a bin of squared cross-section [1].

In this work, it is most convenient to present the Janssen's pressure formula for the case of a rectangular silo of length  $s$  and width  $w$ . In the case of a nearly two-dimensional bin, Fig. 1, we have that  $w \ll s$  and the pressure distribution, under the action of gravity, takes the form

$$p = \rho g \lambda (1 - e^{-z/\lambda}), \quad (1)$$

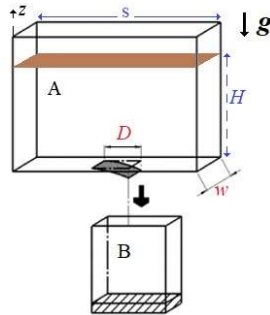
where  $\lambda$  is the characteristic length

$$\lambda = \frac{sw}{2\mu K(w+s)}, \quad (2)$$

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here  $\mu$  is the friction coefficient between the grains and the silo wall and  $K$  is the Janssen constant, which relates the horizontal pressure  $p_h$  to the vertical pressure  $p(z)$ , as  $p_h = K p(z)$ . To use Eq. (1),  $z$  is measured downwards, from the free surface (brown colour zone in Fig. 1) up to the depth  $H$ .

From Eq. (1) we notice that near the surface,  $z < \lambda$ , the pressure is hydrostatic,  $p = \rho g z$ . But, at larger depths  $z > \lambda$ ,  $p \rightarrow p_\infty = \rho g \lambda$ , this means that a large portion of the weight is sustained by the walls.



**Fig. 1.** Schematic of a near two-dimensional bin, A, where an elastic plate of length  $l \sim D$  (very similar in length to the orifice length  $D$ ), is centrally placed at the exit hole. Such a plate is bent by the grains' weight in the bin filled up to a height  $H$ , allowing for a flow of grains which reaches the reservoir B.

## 2.2 Hagen's formula

Hagen [2], around 1852, experimentally studied the quantity of sand flowing, due to gravity, per unit time,  $m'$ , as a function of the diameter  $D$  of a circular orifice centrally located at the bottom of a silo. For  $D$  much larger than the grain size, he found that  $m'$  scales as a power law:  $m' \sim \rho g^{1/2} D^{5/2}$ , *i.e.*, it does not depend on the level of filling, and consequently, is independent of the granular pressure.

In the current case, for our near two-dimensional bin of Fig. 1, several studies [7,8] have shown that if  $D \gg w$ , a scaling of the form  $m' \sim \rho g^{1/2} w^{3/2} D$ , is fairly good.

## 3 Discharge limited by an elastic gate

### 3.1 Large deflection of an elastic plate

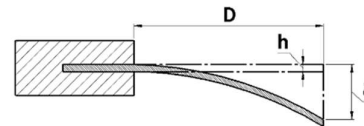
Now, assuming that attached parallel to the rectangular exit hole, of length  $D$  and width  $w$ , lies an elastic plate of length  $l \sim D$  and width slightly smaller than  $w$ . The role of the element is to partially obstruct the gravity driven flow, as seen in Fig. 1. In experiments, every plate used in this study is 1 mm smaller than the corresponding length and width of the hole, in order to avoid contact with any wall surrounding the plate

The fundamental fact in experiments is that there is a pressure  $p_l$  on the plate clamped at an edge of the orifice, see Fig. 2 to look the details of the clamped plate and its deflection. Before filling up the bin, a rigid and

flat element is put in place to support the elastic plate and prevent it from moving. Later on, the granular material is poured evenly over the cross section of the bin in order to ensure uniform grain compaction and uniform pressure distribution at the bottom. Powders which compact fast, are expected to flow easily [9]. Once the silo was filled, the supporting element is suddenly removed causing the deflection of the elastic plate and thus starting the flow.

The physical problem aforementioned fulfils the conditions of the problem to determine the deflection  $\zeta$  of a clamped plate, due to the pressure on it,  $p_l$ , when  $\zeta \gg h$  and  $l \sim D$  is its length. The formal treatment of this problem rather challenging in the context of the theory of elasticity; however, an order of magnitude estimation [6] allows us to find that the deflection is

$$\zeta \sim \left( \frac{l^4 p_l}{E h} \right)^{1/3}. \quad (3)$$



**Fig. 2.** Details of a clamped plate which experiences a large deflection  $\zeta$  due to the grains' weight.

When the local pressure causes a large deflection, it allows the formation of an opening of size  $\zeta$ , and consequently the possibility of a granular gravity flow, which resembles the flow through the sidewalls of silos [10]. In the following Section we shall perform experiments to measure the magnitude of the deflection of the plates and the characteristics of the granular flows across  $\zeta$ .

## 4 Experiments

For the experiments we use an acrylic-made bin which is 25 cm tall, 30 cm deep and 1.40 cm wide, with central exit holes placed at the bottom of the bin and with apertures of lengths  $D = 0.18, 0.19, 0.20, 0.22, 0.24$  and  $0.26$  cm. For all these cases the deflection of acetate plates, acting as the elastic gate, was measured. Commercial acetate elastic plates were used, with Young's modulus  $E = 2.2$  GPa, thickness  $h = 0.125$  mm and lengths close to  $0.18, 0.19, 0.20, 0.22, 0.24$  and  $0.26$  cm.

Ottawa sand used in experiments, consists of round grains, having a  $\rho = 1537$  kg/m<sup>3</sup> bulk density,  $\theta = 31^\circ$  mean value of friction angle,  $\mu_{sand} = \tan \theta = 0.51$  friction coefficient and  $d = 0.5$  mm mean grain diameter. In addition to the use of the elastic plate, experiments were carried out to determine the correlation for  $m' = f(D)$ , *i.e.*, the mass flow rate as a function of the lengths of the orifices. We have found that  $m' \sim 22.4D$ , meaning that there is a linear relation between  $m'$  and  $D$ , as was predicted for near two-dimensional silos [7,8].

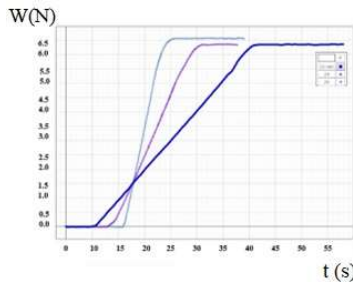
In the cases where the deflection occurs, a granular flow of Ottawa sand is developed, see Fig. 3. In order to assess the nature of this flow we measured the variation of weight over time  $W(t)$  of the flow acting on a force sensor, see Ref. [10]. Recalling that  $W(t)=m(t)g$ , where  $m(t)$  is the time dependent mass. Through the time derivative of the weight, we find the mass flow rate  $m'$ , as

$$m' = \frac{dm}{dt} = \frac{1}{g} \frac{dW}{dt}. \quad (4)$$

In plots of Fig. 4 we show representative measurements of weight evolution for several plates. Due to  $W \sim t$ , we find that for each elastic gate the mass flow rate is a constant.

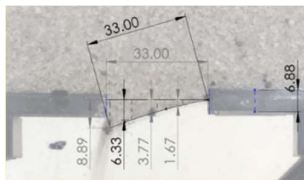


**Fig. 3.** Picture of the granular flow limited by the elastic gate.



**Fig. 4.** Plots of the temporal weight evolution measured for different lengths of the elastic plates, a larger plate allows a larger mass flow rate.

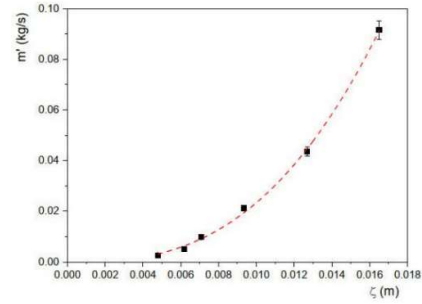
In Fig. 5 we show a measurement scheme of the deflection  $\zeta$ , for a given plate's length,  $D$ .



**Fig. 5.** Detail of the measurement of the plate's deflection.

With this procedure we obtained the plot of the mass flow rate as a function of the deflection, in Fig. 6 we find that data fits a power law of the form  $m' \sim \zeta^{5/2}$ , specifically  $m' = 6449.37 \zeta^{5/2}$ .

The latest result is very intriguing since the Hagen formula, stated for a circular orifice, appears to be valid for the flow limited by the rectangular deflected plate. Thus, we propose a Hagen-like formula in the form



**Fig. 6.** Experimental plot of the mass flow rate as a function of  $\zeta$ . The dashed red curve is the plot of  $m' = 6449.37 \zeta^{5/2}$ .

$$m'_{ph} \sim \rho g^{1/2} \zeta^{5/2}, \quad (5)$$

where  $m'_{ph}$  is now called the phenomenological mass flow rate, and if we directly use the values of  $\rho$  and  $g$  given above, we found that Eq. (5) yields

$$m'_{ph} \sim 4811.60 \zeta^{5/2}. \quad (6)$$

A comparison between the experimental mass flow rate  $m'$ , obtained from experiments in Fig. 6 and  $m'_{ph}$  (Eq. (6)) gives the correlation

$$m' = c m'_{ph}, \quad (7)$$

where the dimensionless constant  $c$  has the value  $c = 1.34$ .

Now, we wish to explore further characteristics of Eq. (5), therefore, we shall use this equation in combination with Eq. (3) for  $\zeta$ , that gives the order of magnitude estimation for the deflection of the plate, then we have that

$$m'_{ph} \sim \rho g^{1/2} \left( \frac{l^4 p_l}{Eh} \right)^{5/6}, \quad (8)$$

we highlight that this last formula establishes a correlation between the flow rate and the local pressure.

In Eq. (8) it is necessary to give the values of the local pressure,  $p_l$ , which were measured for each plate of length  $l$  and width  $w$ . In this case each acetate plate was attached to a plastic plate of the same size, placed at the tip of the force sensor PASCO CI-6537 (with a resolution of 0.1 N). After, pouring the granular material over the whole cross section of the bin until reaching the height  $H = 0.015$  m and to get a uniform pressure at the bottom. By knowing the force and the area of the plate, the local pressure was obtained. In Fig. 7 we show the plot of  $p_l$  vs  $l$ , the red dashed curve accurately follows the relation  $p_l = 43.20 l^{-1}$ , meaning that the local pressure is inversely proportional to length. The previous relation also could be derived by a simple argument: In the bin, at the bottom, the Janssen pressure is  $p$ , given by Eq. (1). This reveals that the net force on the bin floor is  $F = w s p$ , where  $w s$  is the area of the bottom [1]. Also, the force on the plate of length  $l$ ,  $F_l$ , is proportional to the net force

$F$ , meaning that  $F_l = aF$ , where  $a$  is a constant. Additionally, the force on the plate is  $F_l = wlp_l$ , combining all these results we have that  $p_l = a(s/l)p$ , but when  $l \rightarrow s$ ,  $p_l \rightarrow p$  and thus  $a = l$ , and then  $p_l = (s/l)p$ , it means that essentially,  $p_l \sim 1/l$ , which confirms the behavior of the curve of Fig. 7.

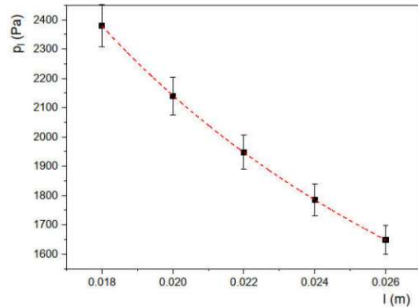


Fig. 7. Plot of the local pressure on the elastic plate as a function of  $l$ .

The use of the multiple values of  $p_l$  of Fig. 7, and of other quantities in Eq. (8), allows us to obtain the plot of  $m'_{ph}$  vs  $l$ , shown in Fig. 8. The curve that fits the data is  $m'_{ph} = 9.18l^{5/2}$ .

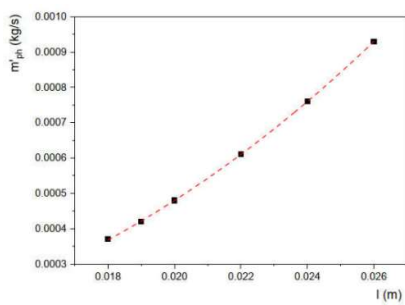


Fig. 8. Plot of the computed mass flow rate, obtained by using Eq. (8), as a function of  $l$ .

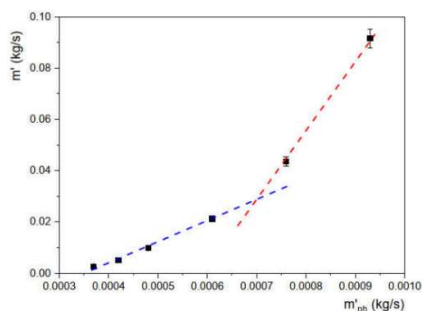


Fig. 9. Plot of the measured mass flow rate,  $m'$ , versus  $m'_{ph}$ , the computed mass flow rate (Eq. (8)). The data were fitted by two different straight lines with different slopes.

In plot 9 we have attempted to correlate the measured mass flow rates  $m'$  to the phenomenological mass flow given by Eq. (8). We observe that the first four data points obey the approximate linear expression  $m' = bm'_{ph}$  where  $b = 79.21$  corresponds to the blue dash line, meanwhile  $b = 282.12$  produces the dash red line. These results seem to highlight the fact that there are two elastic regimes, one valid for large deflections and another one valid for very large deflections.

## 5 Conclusions

In this work we analysed the influence of elastic plates placed at the exit holes of bins, on the mass flow rate,  $m'$ , of cohesionless granular material associated with gravity flow. Experiments show that  $m'$  scales as  $m' \sim \zeta^{3/2}$ . Given this fact, we proposed a Hagen-like correlation in the form of Eq. (5), where the phenomenological mass flow rate,  $m'_{ph}$ , can be computed through the measurements of the deflections,  $\zeta$ , for different lengths of plates,  $l$ . Very similar values of both mass flows can be easily obtained using Eq. (7). Finally, we can carry out an estimate of  $m'_{ph}$  by using Eq. (8), where it is necessary to know  $l$  and the local granular pressure,  $p_l$ . Through the plot of Fig. 9 it was found that the existence of two valid linear elastic regimes is possible, for large and very large deflections.

## Reference

1. H.A. Janssen, Versuche uE uber Getreidedruck in Silozellen. Zeitschr. Vereines Dtsch. Ingenieure **39**, 1045, (1895)
2. G. Hagen, Bericht über die zur Bekanntmachung geeigneten Verhandlungen der Königlich Preussischen Akademie der Wissenschaften zu Berlin. pp. 35–42, (1852)
3. L.D. Landau and I.M. Lifshitz, Fluid Mechanics. Volume 6 of Course of Theoretical Physics, (Pergamon Press: Oxford, UK, 1987)
4. M.A. Aguirre, J.G. Grande, A. Calvo, L.A. Pugnaroni and J.C. Geminard, Pressure independence of granular flow through an aperture, Phys. Rev.Lett. **104**, 238002, (2010)
5. M.A. Aguirre, J.G. Grande, A. Calvo, L.A. Pugnaroni and J.C. Geminard, Granular flow through an aperture: pressure and flow rate are independent, Phys. Rev. E **83**, 061305, (2011)
6. L.D. Landau and I.M. Lifshitz, Theory of Elasticity. Volume 7 of Course of Theoretical Physics, (Elsevier: New Dehli, India, 2015)
7. S. Laidou, P. Ausillous, M. Djermane and B. Dalloz'Dubrujeaud, Discharge flow of granular media from rectangular silos: role of an obstacle and modelling by an orifice at the corner, Mechanics & Industry **21**, 516 (2020)
8. G.M. Brito, A. López-Villa, A. Zacarias, A. Medina, On a general correlation for the discharge rate of grains through slots in thin sidewalls of silos due to gravity, Supl. Rev. Mex. Fis. Rev. **1** (2), 8, (2020)
9. P.G. de Gennes, Reflections on the mechanics of granular matter, Physica A **261**, 267, (1998)
10. A. Medina, D. Cabrera, A. López-Villa and M. Pliego, Discharge rates of dry granular material from bins with lateral exit holes, Powder Tech. **253**, 270 (2014)