

Notes on global permeability – grading curve expressions of granular matter and their grading curve variables

Emőke Imre^{1,2,*}, Mwinkem Delphin Kabey², Ágnes Bálint^{1,3}

¹HBM Research Centre, ²Doctoral School of Applied Informatics and Applied Mathematics, ³Sándor Rejtő Faculty of Light Industry and Environmental Engineering, Óbuda University, Budapest, Hungary

Abstract. This research aims at the elaboration of a globally valid permeability model. A model-building process is made for this purpose. Starting with a 3-variable model, by adding 10 new variables consecutively (into the product-type model generally used at present), an increasingly better fit was attained on a wider database. The last model, with 13 (partly diameter, partly entropy type) variables, was the best, but it needed a “sharp” inverse problem solver. Earlier, only unimodal samples were used on the same database. Series 1 to 4: artificial mixtures of silt and sand grains, $d_{10} = 0.004$ to 0.016 mm, $k = 1.67 \times 10^{-5}$ to 4.76×10^{-3} cm/s. Series 5: sand-gravel $d_{10} = 0.72$ to 5.82 mm, $k = 0.378$ to 50.107 cm/s. Series 9: sand-gravel, $k = 7.0^{-3}$ to 0.48 cm/s. The fit was not perfect on the training database earlier; the present approach is better on a wider database. The permeability variables were started to be analysed. The meaning of two kinds of harmonic mean diameter and d_{10} was examined using fractally distributed grading curves, as a function of entropy coordinate A . Results indicated that d_{10} may vary between the two harmonic mean diameters.

Notations

A	is the normalised or relative base entropy
B	is the normalised entropy increment
C_U	is the coefficient of uniformity ($= d_{60}/d_{10}$)
C_{Cha}	is the “Chapuis’s variable”
C_{Santa}	is the “Ren-Santamarina’s variable”
d	is the grain diameter in mm
d_{10}	is the diameter with 10% of particles are finer
d_h	is the harmonic mean of d by dry mass, $d_h = d_{hm}$
d_{hh}	is the harmonic mean of d^2 by dry mass
d_{hsa}	is the harmonic mean d by surface area
GSD	grain size distribution
k	saturated hydraulic conductivity in cm/s
e	is the void ratio
N	is the number of fractions
r_m	is the hydraulic radius
S_A	is the specific surface area per volume $= S_{sA}$ (the ratio of pore surface area to the bulk volume)
S_0	is the base entropy
ΔS	is the entropy increment

1 Introduction

The Taylor's saturated hydraulic conductivity formula ([2]) contains a term expressing density, an unknown geometric term and an average pore size (volume over surface area of the pores), the hydraulic radius.

Due to the unknown geometric term, slightly modified forms of the Taylor model are widely used (product of variables with calibrated exponent) see e.g.

[1, 3 to 6]). These depend on 2 to 3 variables, are not valid for all granular soils, are non-precise even on the training dataset and cannot be used outside of the training data set (see, e.g., [1, 3]).

To develop a more precise model, we include more than three permeability variables in the same model format and a wide database (Figure 1).

In addition to the traditional diameter variables (e.g. d_{10} and C_U), we use diameter variables reflecting all measured data, such as the d_h or d_{hsa} (harmonic mean diameter based on the GSD by dry mass or by surface area [7], respectively) and the four grading entropy variables [8 to 10]. We use density variables combined with some diameters, from existing models of [5 to 6].

We examine the diameter variables d_{10} , d_h and d_{hsa} using simulated fractal grain size distributions, which found that d_{10} varies between d_h and d_{hsa} as a function of the entropy coordinate A .

2 Grading entropy, fractal distribution

In the grading entropy theory, the *fractions* are defined by successive multiplication with 2, starting from an arbitrary d_0 as follows [7 to 9]:

$$2^j d_0 \geq d > 2^{j-1} d_0 \quad (1)$$

where fractions are numbered by consecutive serial number, $j = 1, 2 \dots n$, see Table 1.

* Corresponding author: imreemok@gmail.com

Table 1. Some defined properties of the fractions, a concept of the grading entropy ($d_0 = 2^{-17}$ mm)

j	14	15	16	17	18
$2^{j-1}d_0$ (mm)	0.0625	0.125	0.25	0.5	1
Entropy $S_{0,i}$	13	14	15	16	17

In a mixture with a fixed j_{min} , the *number of fractions* N between the finest and coarsest non-zero fractions:

$$N = j_{max} - j_{min} + 1 \quad (2)$$

The relative frequencies of the fractions x_i ($i = 1, 2, \dots, N$) for each grading curve fulfil:

$$\sum_{i=1}^N x_i = 1, \quad x_i \geq 0, \quad N \geq 1. \quad (3)$$

Eq (3) describes an $N-1$ -dimensional, closed simplex, where every grading curve is a point with barycenter coordinates x_i ($i = 1, 2, \dots, N$), as a *space of grading curves*. The *grading entropy* S is derived using a uniform elementary cell system with a width of d_0 , assuming a uniform distribution for this within a fraction.

The *grading entropy* S [1] is split into *base entropy* S_0 and *entropy increment* ΔS :

$$S_0 = \sum_{i=1}^N x_i S_{0j(i)} \quad (4)$$

$$S = \sum_{i=1}^N x_i S_{0j(i)} \quad (5)$$

where $j(i)$ is a transformation of the global index.

The *normalised* grading entropy coordinates:

$$A = \frac{S_0 - S_{0,min}}{S_{0,max} - S_{0,min}} \quad (6)$$

$$B = \frac{\Delta S}{\ln N} \quad (7)$$

Let us fix a space of GSD-s. Being a strictly concave function (of the relative frequencies), B has a unique conditional maximum for each $A = \text{constant}$ value:

$$x_1 = \frac{1-a}{1-a^N}, \quad x_i = x_1 a^{i-1} \quad (i > 1) \quad (8)$$

where a is the positive root of the equation:

$$\sum_{j=1}^N a^{j-1} (j - 1 - A(N - 1)) = 0. \quad (9)$$

The *optimal grading curves* have finite fractal distributions, depending on A .

The fractal grading curves are with a central, “*mean*” position among GSDs with fixed A [7 to 9]. They can be used to study a mean behaviour.

The internal stability rule of grading entropy, based on the results of vertical flow tests, is shown in the entropy diagram, for a fixed N value (Fig. 2b). No structure of large grains is present; piping may occur if $A < 2/3$. In the complement zone ($A = 2/3$ and $A > 2/3$), a skeleton of the coarse particles is built up gradually.

Suffusion occurs in the stable zone if more than 2 fractions are missing, if the N -fraction samples may plot below the line $N-2$.

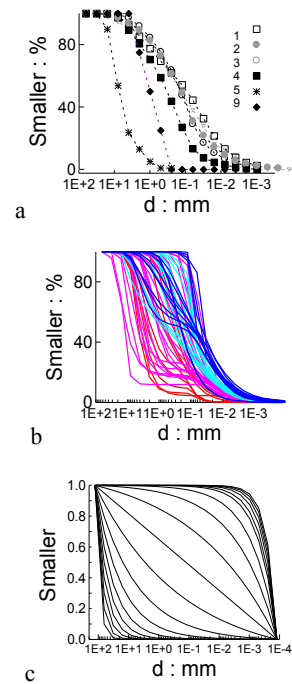


Fig. 1. (a) The previously tested, mean, unimodal GSD-s [1], (b) all GSD-s from series 1 to 4, in blue, light blue, pink, red, respectively. (c) Fractal GSD-s ($N=20$).

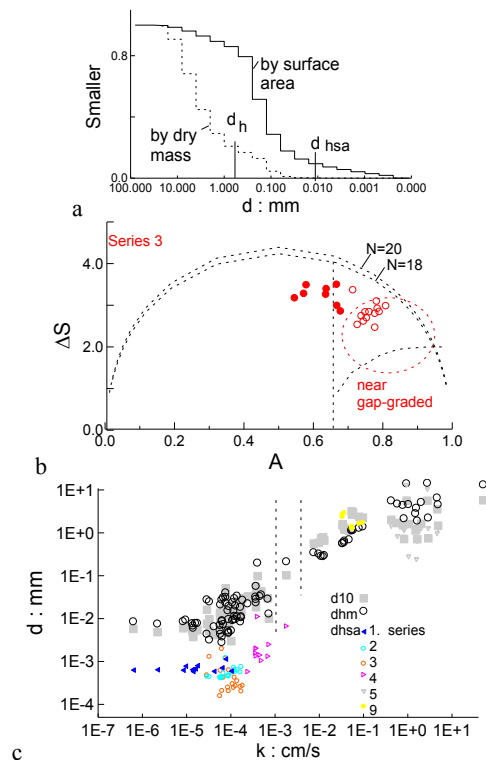


Fig. 2. (a) Typical results for d_h and d_{hsa} (harmonic mean by dry mass and by surface area, respectively) in case of a GSD in series 2. (b) Soil sample series 3 in the grading entropy diagram indicating suffusion zone [8 to 10]. (c) Diameter variables (d_{10} , d_h , d_{hsa}) in terms of k . Dashed lines indicate sample sets removed.

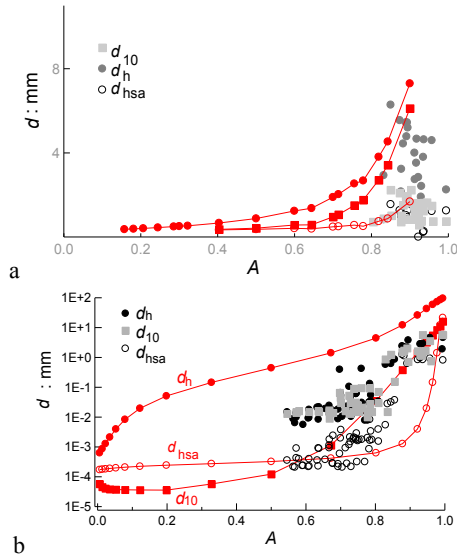


Fig. 3. Comparing d_h , d_{hsa} and d_{10} in terms of A . Computed fractal GSD values in coloured lines. (a) $N=5$ (with series 5 samples), (b) $N=20$ (with most samples).

3 The diameter variables applied

The Taylor formula for spheres is as follows:

$$k = r_m^2 \frac{\gamma_w}{\mu} \frac{e}{1+e} C \quad (10)$$

where r_m is hydraulic radius, γ is permeant's unit weight, and μ is the dynamic viscosity of the permeant, e is void ratio, $e/(1+e)$ is porosity, and the unknown parameter C depends on pore geometry. The formula for the mean pore size, or hydraulic radius r_m , is derived by assuming an N -disperse system, where N is the fraction number, containing the harmonic mean diameter $d_h = d_{hm}$:

$$r_m = \frac{V_v}{S_s} = \frac{e}{6} d_h \quad (11)$$

$$d_h = \frac{1}{\sum_{i=1}^N \frac{x_i}{d_i}} \quad (12)$$

where d_i is any fixed diameter value within a fraction. It is easy to derive the harmonic mean diameter by surface area (d_{hsa}) assuming an N -disperse system, the result is as follows (Fig. 2a):

$$d_{hsa} = \frac{1}{d_{hm} \sum_{i=1}^N \frac{x_i}{d_i^2}} = \frac{d_{hhm}}{d_{hm}} \quad (13)$$

where d_{hhm} is the harmonic mean of d^2 by dry mass. The specific surface area per volume and the ‘‘Ren-Santamarina’s’’ variable [6] are as follows:

$$S_A = \frac{6}{(1+e)d_h} \quad (14)$$

$$C_{santa} = \frac{e^c}{S_A^2} \quad (15)$$

where c is a fitting parameter determined beforehand. The composite ‘‘Chapuis’s variable’’ [5]:

$$C_{Cha} = d_{10}^2 \frac{e^3}{1+e} \quad (16)$$

4 Data analyses

The tested, mean, unimodal [1] and all bimodal GSDs from series 1 to 9 are shown in Fig.1. In this work, bimodal samples were also used.

Series 1 to 4 consist of increasingly bimodal samples, with fixed d_{10} ranges between 0.003 to 0.2 mm, $C_U = 4$ to 482, $N = 13$ to 20, $k = 1.67 \times 10^{-5}$ to 4.76×10^{-3} cm/s. Series 5 is coarse sand – fine gravel, d_{10} : 0.72 to 5.82 mm, C_U : 1.9 to 6.9, $N = 3$ to 9, $k = 0.378$ to 50.107 cm/s. Series 9 is medium sand - fine gravel, $C_U = 1.59$ to 2.15, $d_{10} = 0.28$ to 1.4 mm, $k = 7.010^{-3}$ to 0.48 cm/s, $N = 2$. The 13 variables (S_A , d_{hsa} , d_h , r_m , d_{10} , A , B , e , S_o , ΔS , C_U , C_{Cha} , C_{santa}) were determined for the samples.

The basic diameter variables d_{10} , d_h , d_{hsa} of the samples are shown in terms of k in Fig. 2c. The slope of the imaginary, piece-wise linear regression line (‘‘the k -sensitivity’’) was similar for d_{10} , d_h being both different for small k ($< 10^{-4}$ cm/s) and in the complement zone. The bimodal, outlier samples were removed; they were identified in the suffusion zone, as shown in Fig. 2b.

The less-known d_{hsa} is related to a kind of minimum grain size, and showed a scatter in the $k < 10^{-2}$ cm/s range depending on the sometimes large C_U (Fig. 2a,c).

To assess the ‘‘mean’’ behaviour for variable d_{10} , some fractal grading curve series were simulated in terms of A for $N=5$ and 20 (Fig. 1c). The d_{10} , d_h and d_{hsa} were determined, and compared with real sample data. Fig. 3 presents the computed fractal variables d_{10} , d_h and d_{hsa} in terms of A , for $N=5$ and 20. For large A ($A > 0.8$), the d_{10} was close to d_h , and for small A ($A < 0.6$), it was close to d_{hsa} . Concerning the samples, $d_h > d_{10} \sim d_{hsa}$ and $d_h \sim d_{10} > d_{hsa}$ were met, for $k > 10^{-2}$ cm/s and for $k < 10^{-2}$, respectively.

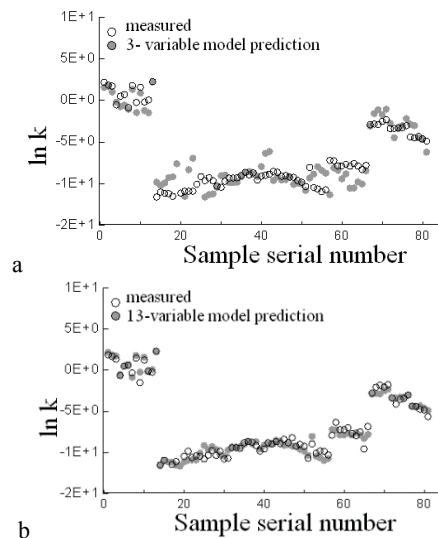


Fig. 4. Measured and fitted data by eye, 80 training samples. (a) the start: the $S_o - \Delta S - e$ model. (b) the last, 13-variable model (S_A , d_{hsa} , d_h , r_m , d_{10} , A , B , e , S_o , ΔS , C_U , C_{Cha} , C_{santa}).

Table 2. Correlation coefficient R of measured - computed data, in a function of the parameter number

parameter number	R	note
14	0.9871	best
13	0.9865	
9	0.981	NE
6	0.968	NE
4	0.958	NE

NE= without grading entropy parameters

5 Model-building process

In the frame of a model-building process, some model series with an increasing number of variables were generated in the form:

$$k = a d_{\text{hsa}}^c e^h C_u^k d_{10}^l \quad (17)$$

where a , b , c , h , k , l are unknown parameters, determined from measured data by general linear inverse problem solution (see, eg, [11]). Using all 13 variables (S_A , d_{hs} , d_h , r_m , d_{10} , A , B , e , S_0 , ΔS , C_U , C_h , S_{anta}) and 14 parameters, the resulting model was the best, but needed a sharper inverse problem solver (Fig. 4).

6 Discussion, conclusion

The results are discussed and concluded as follows.

(i) The diameter variables d_h and d_{hsa} were derived assuming N -disperse systems, following the general practice. The effect of the discretization (ie., here the N fractions) and the way of the selection of d_i in the fraction, may need some further research.

(ii) The harmonic mean diameter by dry mass d_h is related to the hydraulic radius r_m , which is a mean pore size. The harmonic mean diameter by surface area d_{hsa} is “small” (Fig. 2a), can likely be related to a kind of minimum pore size, depending on d_h and C_U [7].

(iii) Representing the d_{10} and d_h in terms of k , the slope of an imaginary, piece-wise linear regression was significant at $k > 10^{-4}$, and small otherwise, where some additional variables are needed in the k -modelling. The outliers - due to suffusion- were excluded (Fig. 2).

(iv) In terms of fractal grading curves uniquely depending on A (Figs.1c, 3), the d_{10} varied between d_h and d_{hsa} . The d_{10} was similar to d_h for $A > 0.8$ and to d_{hsa} for $A < 0.6$, indicating that d_{10} is can be an independent variable in the k - modelling.

(v) For the samples, $d_{10} \sim d_{\text{hsa}}$ and $d_h \sim d_{10}$ were met at $k > 10^{-2}$ cm/s and at $k < 10^{-2}$, resp. To compare real and fractal variables, further investigation is needed since sample fraction numbers are $N = 2$ to 20 and only fractals distributions with $N = 5$ and 20 were considered.

(vi) A preliminary model-building process was successfully made. Starting from a 3-variable model [1], by

increasing the variable number, the fit became gradually better on the training data set, and using all 13 variables, the fit was the best (Fig. 4).

The number of variables in the model-building process had a numerical limit. The mathematical analysis of the problem, including the maximisation of the number of valid digits in the identified model parameters requires further research [11].

References

1. E. Imre, Z. Illés, et al, Grading curve relations for saturated hydraulic conductivity of granular materials. *Environmental Geotechnics*. 1–85 (2025) <https://doi.org/10.1680/jenge.23.00131>
2. D. W, Taylor, *Fundamentals of soil mechanics*. (John Wiley & Sons, Inc., New York, USA, 1–700, 1948).
3. B. C. O’Kelly and M. Nogal, Determination of soil permeability coefficient following an updated grading entropy method. *Geotechnical Research*, **7(1)**, 58-70 (2020) <https://doi.org/10.1680/jgere.19.00036>
4. S. Feng, D. Barreto, E. Imre, E. Ibraim & P.J. Vardanega, Use of hydraulic radius to estimate the permeability of coarse-grained materials using a new geodatabase. *Transportation Geotechnics*. **41**, 101026 (2023) <https://doi.org/10.1016/j.trge.2023.101026>
5. R. P. Chapuis, Predicting the saturated hydraulic conductivity of sand and gravel using effective diameter and void ratio. *Can Geotech J.* **41(5)**,787–795 (2004) <https://doi.org/10.1139/t04-022>
6. X. W. Ren and J. C. Santamarina, The hydraulic conductivity of sediments: A pore size perspective, *Engineering Geology*. **233**, 48–54 (2017) <https://doi.org/10.1016/j.enggeo.2017.11.022>
7. L. Trani, B. Indraratna, The use of particle size distribution by surface area method in predicting the saturated hydraulic conductivity of graded granular soils. *Geotechnique* **60(12)**, 957–962 (2010) <https://doi.org/10.1680/geot.9.T.014>
8. E. Imre, V.P. Singh, The Finite Fractal Distributions as Mean Grain Size Distributions of Granular Matter. *Fractal Analysis*, Chapter 7, (2024) <https://doi.org/10.5772/intechopen.1003760>
9. E. Imre et al, Statistical parameters and grading curves In: *Proc of the 20th ICSMGE*, Sydney, (2022) 5, 302 p. 713-718. https://www.issmge.org/uploads/publications/1/12/0/ICSMGE_2022-122.pdf
10. J. Lőrincz, *Grading entropy of soils*, Doctoral Thesis, TU of Budapest, 1986 (in Hungarian). <http://hdl.handle.net/10890/60401>
11. Cs. J. Hegedüs, Reorthogonalization Methods Revisited *Acta Polytechnica Hungarica*. **12(8)**, (2015) https://acta.uni-obuda.hu/Hegedus_64.pdf