Almost-off-the-shelf tools for ICRH modelling

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Abstract. The present paper shows a number of numerical methods under development at LPP-ERM/KMS to model wave propagation and damping in the RF domain of frequencies. The philosophy is to adopt a maximum of physics tools already available in the literature (i.e. *no* new derivations are proposed; at most mild, intuitive upgrades of the existing analytical description are introduced) and extend them to make the description more realistic while keeping the CPU needs as modest as possible.

1 Introduction

Studying the dynamics of the wave-particle interaction underlying Ion Cyclotron Resonance Heating in tokamaks or stellarators requires the development and solution of the relevant wave equation. Over the last decades, a large number of authors (see e.g. [1, 2] and references therein; particularly noteworthy are the historical and visionary efforts of Kaufman [3], formulating a general approach to describe the wave-particle interaction still not fully accounted for in all its levels of sophistication even today) have contributed to pushing the realism of the models forward so a lot of detailed expressions - with various degrees of fidelity while concentrating on specific subtopics, each deserving attention - are available for exploitation. Crudely speaking, there are 2 main challenges when modelling wave heating in magnetic confinement devices while attempting to capture key aspects of the dynamics of the wave-particle interaction of RF created or fusion-born fast particle populations: (i) the perpendicular dynamics is typically described in terms of guiding center variables and requires accounting for finite temperature (finite Larmor radius) effects, and (ii) as the magnetic field cannot act on the particles along its field lines, the impact of the parallel motion on the wave-particle interaction is generally less localized than the perpendicular motion; when the magnetic field varies along the orbit, the associated modification of the dielectric response needs to be accounted for. Similarly, gradually more sophisticated numerical tools become available for solving partial differential equations in complex geometries. Combining the strength of both theoretical efforts and scientific libraries available offers perspectives for the development and exploitation of "almost-off-the-shelf" tools with high degree of efficiency as well as richness of physics. The present contribution summarizes - nonexhaustively - work ongoing at LPP-ERM/KMS aimed at adopting this mixed-tool approach while equally sketching a way forward to further improve the description. In particular, it is the somewhat more pen-and-paper counterpart of the contribution by V. Maquet [4] at this conference, the latter putting the focus on the numerical implementation. The interested reader can find details on the mathematical formulations of the methods in the provided references; the present conference paper merely shortly sketches the adopted philosophies and provides a few illustrations. A second line of the LPP-ERM/KMS research places the accent on pushing the analytical developments further when accounting for the plasma inhomogeneity; see [1, 20] for details on those aspects.

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2 General notes

The finite element method is an often exploited technique to solve differential and integro-differential equations. The basic idea is to turn finding the solution of the relevant equation into finding the function that minimises a variational form. The equation is multiplied by a test function and integrated over the domain of interest. Then the domain of interest in subdivided into finite sized elements in which the desired function is - most often - locally approximated by a fairly low order polynomial, allowing to evaluate some of the needed integrals by hand. In variational form, the wave equation to be solved can be written

$$\int d\vec{x} \int d\vec{k} \int d\vec{k'} \vec{F}^*_{\vec{k'}} . \overline{\overline{G}}(\vec{k}, \vec{k'}, \vec{x}) . \vec{E}_{\vec{k}} exp[i(\vec{k} - \vec{k'}).\vec{x}] = \vec{0},$$

in which $\vec{F}$ is the test function vector and $\vec{E}$ is the electric field. Starting from Ichimaru's original description [5], but adding in knowledge from other work (e.g. [2, 3, 6, 7], the relevant wave equation governing the propagation and damping in the RF domain of frequencies is

$$\int d\vec{x} [k_o^2 \vec{F}^* . \overline{\overline{\epsilon}} . \vec{E} - (\nabla \times \vec{F})^* . (\nabla \times \vec{E})] = - \int d\vec{S} . \vec{F}^* \times (\nabla \times \vec{E}),$$

where

$$\vec{F}^*.\bar{\bar{\epsilon}}.\vec{E} = [1 - \sum_\alpha \frac{\omega_p^2}{\omega^2}\vec{F}^*.\vec{E}] - 2\pi \sum_\alpha \frac{\omega_p^2}{\omega^2} \sum_{N=-\infty}^{+\infty}$$

$$\int_0^\infty dv_\perp \int_{-\infty}^{+\infty} dv_{//} \int d\vec{k} \int d\vec{k}' exp[i(\vec{k} - \vec{k}').\vec{x}]$$

$$\frac{[N\Omega\partial F_o/\partial v_\perp + k_{//,mean} v_\perp \partial F_o/\partial v_{//}]}{[N\Omega + k_{//,mean} v_{//} - \omega]} [\mathcal{L}_{\vec{k}'}(\vec{F}_{\vec{k}'})]^* [\mathcal{L}_{\vec{k}}(\vec{E}_{\vec{k}})],$$

where $k_{//,mean} = [k_{//} + k'_{//}]/2$, $\mathcal{L}$ is the Kennel-Engelmann operator [8] and F_o is the slowly varying distribution function in normalised form ($\int d\vec{v} F_o = 1$; the density factor is embedded in the ω_p^2 factor). The labels "parallel" and "perpendicular" are with reference to the confining magnetic field. The term on the right hand side is a surface term corresponding to the flux when writing down the conservation law associated with the equation (which is obtained by replacing the test function by the electric field). The jump of that flux at the antenna can be introduced as an internal boundary condition. Ichimaru's expression implicitly assumes adiabatic switching on of the electric field along the path of charged particles in a strong magnetic field, an assumption that is not necessarily verified and will be re-assessed later (see sections 6 and 7). Its domain of applicability also depends on whether the dependencies of the slowly varying distribution function are accounted for. Many authors have been devoting work on these aspects over many years. Any expression exploited to date also implicitly assumes the gradients of the electric field are stronger than those of the equilibrium quantities, allowing some of the less important curvature effects to be omitted while keeping the dominant ones. In particular, when making transformations one should account for the fact that the projection vectors themselves have a spatial dependence and hence there is a difference between the gradient of a vector $\nabla\vec{v} = \nabla \sum_i [v_i \vec{e}_i]$ and $\sum_i [\nabla v_i]\vec{e}_i$ (where $v_i = \vec{e}_i.\vec{v}$ with the 3 $\vec{e}_i$ spanning the 3-dimensional space), the distinction involving the metric Christoffel symbols.

The accent in the present paper is primarily on illustrating opportunities for application of existing numerical techniques; the plasma model used can be further generalised later. $\vec{E}$ is the electric field and $\vec{F}$ is the test function vector. The Poynting flux is continuous and hence surface terms such as the ones on the right hand side can - to the exception of the antenna location - be omitted at first sight. Whether the surface term needs to be included in the variational formalism depends on the continuity of the adopted base functions: if the numerical expression jumps across finite elements then the surface term is needed to guarantee the physical flux it describes is indeed continuous. In the same vein extra terms of the kinetic flux need to be included when the plasma admits finite temperature correction modes that do not survive in a cold plasma or in vacuum (while the Poynting flux does). In k-space explicit expressions can be found for a variety of distributions. For example, the result for a given $\vec{k}$ and $\vec{k}'$ of the electric field and the test function vectors, respectively, the $K_{\perp 1,\perp 1}$ coefficient obtained for the case the distribution F_o is Maxwellian is

$$2\pi \sum_\alpha \frac{\omega_p^2}{\omega^2} \sum_{N=-\infty}^{+\infty} \int_0^\infty dv_\perp \int_{-\infty}^{+\infty} dv_{//} \frac{v_\perp F_o}{v_t^2}$$

$$\Big[1 + \frac{\omega}{[N\Omega_\alpha + k_{//} v_{//} - \omega]}\Big] G_{N,1}(\vec{k}) G^*_{N,1}(\vec{k}') =$$

$$\sum_\alpha \frac{\omega_p^2}{\omega^2 v_t^4} \sum_{N=-\infty}^{+\infty} \Big[1 + \frac{\omega}{k_{//} 2^{1/2} v_t} \mathcal{Z}_N\Big] e^{iN[\psi - \psi']}$$

$$[\frac{(N\Omega)^2}{k_\perp k'_\perp} \cos\psi \cos\psi' \mathcal{I}_1(\nu, a, b, p) + \frac{iN\Omega}{k_\perp} \cos\psi \sin\psi' \mathcal{I}_2(\nu, b, a, p)$$

$$- \frac{iN\Omega}{k'_\perp} \sin\psi \cos\psi' \mathcal{I}_2(\nu, a, b, p) + \sin\psi \sin\psi' \mathcal{I}_3(\nu, a, b, p)], \quad (1)$$

in which $k_\perp$ is the magnitude and ψ the angle between a reference perpendicular direction and the perpendicular part of the wave vector (and similar for $k'_\perp$ and ψ') [9]. The essential dynamics of the wave-particle interaction near the Nth cyclotron resonance $\omega = N\Omega$ is captured by the Fried-Conte function $\mathcal{Z}$ with argument $[\omega - N\Omega_\alpha]/[k_{//} 2^{1/2} v_t]$ and in which the functions $\mathcal{I}_\alpha$ describe the perpendicular dynamics originating from terms proportional to $v_{//}^\alpha$ in the numerator of the double product of the Kennel-Engelman operator.

3 The Budé method

Budé proposed to approximate the coefficients of the dielectric tensor expressed in $\vec{k}$-space by polynomial fit, allowing to make the inverse Fourier transform in a trivial way. The original work by Budé [10] made the proof of principle and established the validity of the method: he demonstrated that the integro-differential operator can be approximated by a sufficiently high polynomial fit. Subsequent work [6] removed a weakness of the original paper: aligned with the outcome of a discussion on what the proper interpretation of "plasma heating" is in a medium of moving charged particles with which electromagnetic waves interact and that was settled in the 80's (see e.g. [11]) a proper placement of the differential operators is required to ensure the positive definiteness of the absorbed power in case the plasma is in thermodynamical equilibrium, as is expected fro first principles. Thanks to the powerful formulation of Ichimaru, expressions for the dielectric tensor operator for the cases of Maxwellian as well as bi-Maxwellian distributions with a parallel drift were obtained. At the expense of extra computational effort needed to evaluate the expressions, the tensor for arbitrary distributions - typically provided by a Fokker-Planck solver on a grid in velocity space - was also obtained. The Budé technique was successfully exploited in 1D and its potential is now being explored in 2D (see e.g. [4]). Figure 1 shows the perpendicular electric field component excited by a (simplified) fast wave antenna in a JET-type $(H) - D$ discharge heated by minority cyclotron heating at $f = 51MHz$ for $B_o = 3.45T$ adopting a "tepid" plasma model (i.e without finite Larmor radius corrections but including the thermal effects in the parallel direction) as predicted using the NGsolve software [12]; the plot shows

the $k_{//}$-up or downshift due to the presence of the poloidal field for 3 different plasma currents I_p, causing the different bending of the wave front for different I_p.

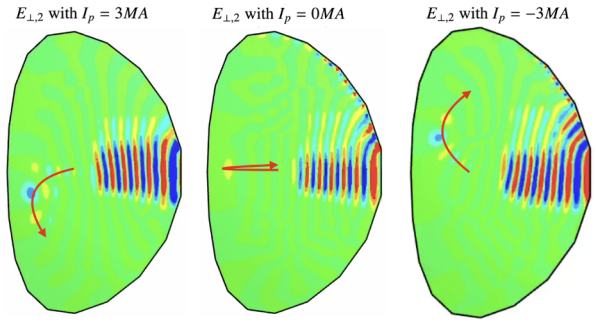

Figure 1. 2D "tepid" plasma wave equation solution using the NGsolve software and for 3 different plasma currents: $I_p = 3MA$ (left), $I_p = 0MA$ (middle), and $I_p = -3MA$ (right).

4 The fast-AORSA method

Jaeger [13] solved the integro-differential equation governing the wave propagation and damping relying on Fourier analysis. He approximated the continuous Fourier integral in $\vec{k}$-space by a discrete set of modes. Imposing the wave equation at a number of grid points the wave equation is replaced by a massive system of coupled linear equations through which he identified the Fourier components of the electric field. The advantage of this technique is that the wave equation is formally simple and a minimum of algebra is needed. The drawback is that the resulting linear system is huge so that inverting the system matrix requires a lot of memory and CPU time. The size of the system is extremely large since the oscillatory Fourier factors $exp[i\vec{k}.\vec{x}]$ never decrease in amplitude while capturing the wave dynamics requires a sufficiently refined grid both in physical and $\vec{k}$-space. Exploiting a mixed Fourier-finite element technique allows to drastically cut back on the required CPU time: multiplying the equation with a test function vector and integrating it over a finite interval achieves 2 simplifications: (i) the oscillatory behaviour can be captured more accurately or with more precision by semi-analytically integrating the local wave equation, and (ii) the weak coupling expected between distant Fourier modes in $\vec{k}$-space is exploited since the oscillation over a full period yield a near-zero result: rather than the oscillating factor Jaeger adopted, the 2 oscillating factors of the Fourier components of the electric field and the test function vector now appear, and hence the integrals that need to be evaluated are of the form $\int_a^b dx_\alpha exp[i(k_\alpha - k'_\alpha)x_\alpha]G(k_\alpha, k'_\alpha, x_\alpha)$ in each direction, in which G is a function that - provided a suitable grid is used - locally can be approximated by a polynomial. The fast-AORSA technique was successfully exploited in 1D [7] and its potential is now being explored in 2D [14]. Figure 2 provides the solution of a simple 2D wave equation in a rectangular box adopting the fast-AORSA technique. The wave is excited via a boundary condition on the right edge of the physical boundary (represented by the red dashed lines) inside the actual integration domain using a finite E_Z there. Otherwise metallic boundary conditions are imposed on the physical edges. As the electric field is written as a discrete sum of Fourier modes $exp[i(k_R R + k_Z Z)]$ - forcing it to be periodic across the edges of the integration domain - while the solution itself is not, an artificial non-physical domain is added to the integration integration domain to ensure periodicity.

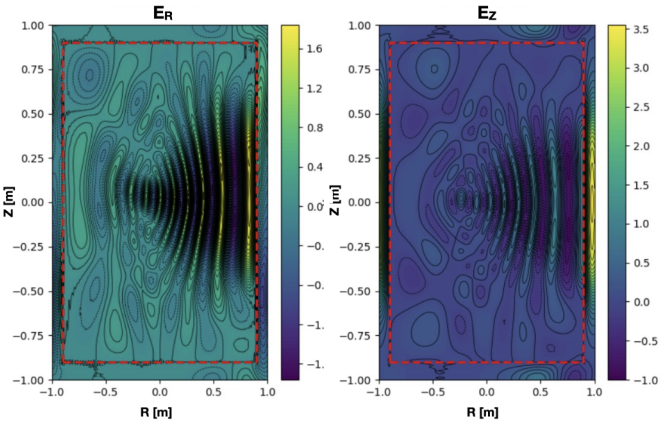

Figure 2. Example of wave equation solving using the 2D version of the fast-AORSA approach.

5 The quasi-base method

Ever since it became clear that adopting the guiding centre position rather than the particle position drastically simplifies the description of the interaction between electromagnetic waves and charged particles on their forced gyro-orbit imposed by the strong confining magnetic field, authors have preferred to recast the wave equation in $\vec{k}$-space and to locally simplify the description by adopting a "quasi-homogeneous" approach. Many variants of this exist, and with varying degree of realism. Although the Fourier representation strictly speaking is a perfectly suitable representation, it is in practice usually preferred to write the actual wave equation in $\vec{x}$-space and use the finite element technique to solve it. The fast-AORSA technique exploits the philosophy of the finite element approach to recast the equation solved by Jaeger into a form that eliminates unneeded terms but hardly profits from the wide variety of finite element numerical tools nowadays available. An option to tap into this potential is to perform the integrals in $\vec{k}$-space by first finding the Fourier transform of the finite element base functions [15].

We illustrate the philosophy in 1D: one expects the physical spectrum of modes to be accounted for has a specific width. The grid of finite elements is based on that i.e. the chosen base functions allow to capture the shortest expected wavelengths or evanescence lengths. The finite elements locally allow to write the electric field (and similarly the test function) by $E_\alpha(x) = \sum_i E_{\alpha,i} B_i(x)$. The B_i can for example be the Hermite polynomials in which case the $E_{\alpha,i}$ coefficients are the electric field values and their derivatives at the grid points delimiting each finite interval. Combining these 2 by 2 in 2 neighbouring elements

$[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$ we can form 2 "hat" functions: both are zero and have a zero first derivative at the edges x_{i-1} and x_{i+1} while the first has a peak value 1 at x_i and the second has a peak value 1 of its derivative; both are smooth and have smooth derivatives at that mid point. The plasma term is formulated in terms of Fourier components i.e. we have terms of the form

$$\int dx \int dk \int dk' F_{k'}^* G(k,k',x) E_k exp[i(k-k')x].$$

Introducing the base functions we have

$$E_k = \sum_i E_i \int dx exp[-ikx] B_i(x) = \sum_i E_i B_{i,k}$$

and since the base function is only finite in a small domain we can find that integral by hand. Embedding B_i in a sufficiently wide domain so that the function is zero at both ends we can write it in terms of a suitable Fourier sum rather than a Fourier integral. The earlier introduced hat functions cover 2 elements of width Δx (grid size). Adding N_{extra} elements on each side we get a total extent $2(N_{extra}+1)\Delta x$. Introducing the Fourier angle θ ranging from $-\pi$ to $+\pi$ we can then find the Fourier components of the hats. To be able to describe the structure inside the 2 intervals where the "hat" is nonzero we need more than just the mode that fits inside Δx. Taking N_{detail} extra points in that interval to manage that we then have $2(N_{extra}+1)N_{detail}$ intervals so the same number of basic modes and total grid points of the subgrid. This yields the spectrum $-(N_{extra}+1)N_{detail}+1 \le m \le (N_{extra}+1)N_{detail}$. The maximal k then satisfies $k_{max}2(N_{extra}+1)\Delta x = m_{max}2\pi$ so that $m_{max} = (N_{extra}+1)N_{detail}$. For an imposed k_{max},

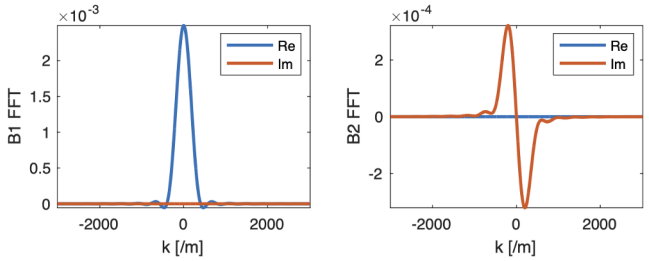

Figure 3. Fourier spectrum of the 2-by-2 Hermite base functions.

Δk and Δx we can find the optimal N_{extra} and N_{detail}: $N_{extra} = \frac{\pi}{\Delta k \Delta x} - 1$ and $N_{detail} = \frac{k_{max}\Delta x}{\pi} = \frac{k_{max}}{[N_{extra}+1]\Delta k}$. For a given Δx, the k_{max} needs to be slightly adjusted for the above to be integer. Figure 3 provides the Fourier spectrum of the 2-by-2 combined Hermite polynomials. Filling this out in the relevant integral we get terms of the form

$$\sum_{i,i'} E_i F_{i'}^* \int dx \int dk dk' B_{i',k'} * G(k,k',x) B_{i,k} exp[i(k-k')x] \approx$$

$$\sum_{i,i',j,j'} E_i F_{i'}^* (\Delta k)^2 B_{i',k'}^* G(k_j, k_{j'}, \tilde{x}) B_{i,k} \int_{x_i}^{x_{i+1}} dx exp[i(k_j - k_{j'})x],$$

in which $\tilde{x}$ is the mid point in the (sufficiently fine) grid and the integrals that appear are of the same form as the one we needed to evaluate to get the (quasi-) Fourier representation of the base functions. If needed or wanted, the spatial inhomogeneity of the G can easily be accounted for by finding the G_i that expresses the G in terms of the adopted base. The first 2 sums in the above already naturally appear in the finite element formulation but the latter 2 represent the detail of the dielectric response. All "building blocks" are computed once and for all i.e. all needed integrals can be written in the form of vector-matrix-vector multiplications. Note that the Fourier base is not an actual base since it does not cover the whole x-range. We refer to it as a quasi-base. Note also that - opposite to the original Hermite polynomials - the base functions are continuous and have continuous derivatives so that they connect continuously across the grid points.

6 Generalising the integrals

Ichimaru's expressions only contain information of the present time. In practice, inverting Vlasov's equation brings the time evolution of the wave-particle interaction along the whole history of the trajectory. Generalising the factor $exp[i(\vec{k}-\vec{k}').\vec{x}]$ appearing in the variational form of the wave equation, the integrals needed to find the dielectric response now contain the factor $exp[-i\tilde{\Phi}(t)] \int_{-\infty}^{t} dt' exp[+i\Phi(t')]$ (strictly, a slowly varying function with other dependencies needs to be added but since those variations are much more modest and the contributions will need to be evaluated in tiny finite element intervals or volumes, this is of secondary importance). Here the tilde refers to the term (and $\vec{k}$) in front of the integral and that without it to the one under the integral. We adopt t_o as the - to be defined - reference time and make a series development of the phase; we introduce the notations $T = (t'+t)/2 - t_o$ and $\tau = t' - t$ and subsequently adopt the choice that t_o is the mean time $(t+t')/2$ i.e. that $T = 0$. Two types of expressions are adopted in the literature: either one only makes a Taylor series development of Φ or - more symmetrically and corresponding to more general theory (Kaufman [3] being the earliest while also most basic one, and e.g. Bécoulet [16] and Lamalle [2] offering practical expressions) - of both Φ and $\tilde{\Phi}$. We will henceforth adopt the notation $\overline{\Phi}_0 + \overline{\Phi}_1\tau + \overline{\Phi}_2\tau^2$ for either of the cases. The factors independent of time are the $\vec{k}.\vec{x}$ at the reference position. The factors that are linear in time are the usual factors $N\Omega + k_{//}v_{//} - \omega$ (where either $k_{//}$ is the parallel wave number of the field or the mean $k//$ of the test function vector and the electric field) appearing in the resonant denominator in homogeneous theory. Finally the last factor describes the acceleration or deceleration due to the change of Ω and $v_{//}$ along the guiding centre orbit. It speaks for itself that the Taylor series expansion only holds for times and positions close to the reference time and position. Significant changes of these parameters require a piece-by-piece evaluation of the time integral when integrating backward in time. After 2 changes of variables one gets the integral

$$exp\Big[i[\Phi_0 - \tilde{\Phi}_0]\Big] \frac{\pi^{1/2}}{2\alpha} Erfc(\beta) exp[\beta^2]$$

in which $Erfc$ is the complementary error function and where

$$\alpha = e^{-i\pi/4}\overline{\Phi}_2^{1/2}, \beta = i\overline{\Phi}_1/[2\alpha].$$

This smooth function is only varying significantly when the absolute value of its argument is below 2 and it can be considered as a constant elsewhere i.e. the wave-particle response it models is well localised near the resonance where Φ_1 crosses zero. Note that the argument of the function diverges when the acceleration term vanishes. In most cases, this divergence is an artefact without consequence, however: when the quadratic term can justifiably be neglected w.r.t. the linear term (i.e. when $\beta \to \infty$ so that $Erfc(\beta) = 1/[\pi^{1/2}\beta exp[\beta^2]]$) the expression reduces to the familiar homogenous plasma resonant denominator

$$\frac{1}{i\Phi_1} exp\Big[i[\Phi_0 - \tilde{\Phi}_0]\Big] = \frac{exp[i(\vec{k'} - \vec{k}).\vec{x}]}{i[N\Omega + \mathbf{k}_{//}v_{//} - \omega]}$$

(with $\mathbf{k}_{//}$ either being $k_{//,\vec{E}}$ or $k_{//,mean}$). Note that the latter completely discards past events and merely retains the contribution from the last time t in the path integral. Since the phase factor depends on the local quantities, it makes sense to replace the integral defining the time history by a sum of contributions $\sum_i exp[-i\tilde{\Phi}(t)] \int_{t_i}^{t_{i+1}} dt' exp[+i\Phi(t')]$ in which the last t_i is t and the sum extends over the part of the path adding significant contributions.

7 Nonlocal response

So far the path integral was treated using the usual approximation of "adiabatic switch-on" of the electric field which allows to assume only the contribution at the present time t or at most some small time interval before it needs to be accounted for. In practice there is no guarantee that that is justified: the electric field is always a local quantity but the wave-particle interaction is not. To account for the full time history it suffices to split $\int dt'$ into small parts and to sum on them. Adopting the finite element approach, local approximations exploiting low order polynomials is standardly done so - provided a suitable coordinate is used to enable accurately capturing the phase variation along the (guiding centre) orbit - these supplementary integrals can be evaluated using the base function integrals already in place. In the case of an axisymmetric tokamak the constants of motion are known and the orbits are poloidally closed. Because of the finite poloidal field the orbit only rarely closes on itself toroidally, however. Adopting a zero orbit width approximation - so that the guiding centres simply follow field lines - the latter exceptional cases occur on "rational surfaces" where the field lines close on themselves. Rather than needing to track the time history for each individual subinterval, the evaluations already performed when assembling the building blocks for solving the wave equation and transforming it into a linear system to be solved suffice: after a full poloidal turn and subtracting the (integer) number of times the phases has changed by 2π, the effective phase change is $\Delta\Phi$; this phase change is for a single orbit, poloidal and toroidal mode number as well as plasma current and is not representative for a whole distribution. The previous transit is identical to the first one i.e. a common factor can be shifted in front of the integral. Only very exceptionally this phase change is an integer fraction of 2π (in which case the energy kicks particles receive at each crossing of a resonance are simply cumulative so that a strong wave-particle interaction results) and hence accounting for the infinite amount of poloidal transits to cover the times from $t' = -\infty$ to $t' = t$ much more typically yields the various phases at each crossing fill the interval from 0 to 2π randomly:

$$\sum_{M=0}^{\infty}[...]exp[iM\Delta\Phi] = [...]\sum_{M=0}^{\infty} exp[iM\Delta\Phi].$$

That would mean that the net interaction (in particular the energy kick) of the wave-particle interaction has the same amplitude but has a random phase $exp[iM\Delta\Phi]$ so that the sum is simply zero when properly accounting for the whole history along the path. This situation is known as "superadiabaticity" [17] but is - luckily - not the standard one: in practice the very small but finite amount of collisions (or more generally any decorrelation mechanism) makes the difference. Strictly one should upgrade Vlasov's equation that yields the perturbed distribution function in terms of the slowly varying one by adding the impact of the collision operator. This is a daunting task. In practice simplified models are often adopted (see e.g. [18]) to get a first approximate view. The simplest among these is the Krook operator, which modifies the equation of motion by adding a small damping term. The final expressions are identical to the ones standardly exploited except for 1 small - but key - detail: the real driver frequency ω is supplemented with a small but finite imaginary part ν (the collision - or more general decorrelation - frequency). Labeling the transition time needed for a full poloidal transit as τ_t the integral then not only has a phase change $\Delta\Phi$ but also an amplitude change $exp[\nu\tau_t]$ after each transit. Summing over all past transits towards $-\tau_t$, $-2\tau_t$, ... now yields a converging sum. For the Krook operator this yields a multipass factor $\sum_{M=0}^{\infty} exp[M(i\Delta\Phi - \nu\tau_t)]$ as illustrated in Fig.4. For very fast decorrelation (very short decorrelation time e.g. a highly collisional regime) the multipass factor does not change at all but the integral is finite; in that regime the exact value of the decorrelation time does not matter (which is the justification for the "collisionless" limit where one accounts for causality to compute the proper behaviour when crossing the resonance but does not worry about setting the decorrelation frequency right). For very long decorrelation time, we recover the - unwanted - case where there is no net interaction. Again the details of the wave-particle are irrelevant. The details of the interaction *do* matter when the transit time is of the order of the decorrelation time. This simple computation illustrates how the small but finite collisionality is crucial to justify the "memory loss" invoked to isolate resonances. A somewhat more sophisticated study - e.g. accounting for various decorrelation models - was made in [2, 19] and work is ongoing to include the associated dynamics. Note that to obtain a fully proper expression for $\Delta\Phi$ one

needs to go back to the level of individual orbits: as long as the history of a particle is unimportant one can locally integrate over velocity space but adding extra sophistication resulting from past event requires properly identifying which local velocity corresponds to which velocity in the neighbouring finite elements, the particle energy as well as the magnetic moment being conserved but not the parallel and perpendicular velocity. The most direct way to do this is to replace time as independent variable by arclength along the (guiding centre) orbit. For the simple case of the axisymmetrical tokamak and omitting drift corrections one can write $dt = d\theta/\dot{\theta}$ where $\dot{\theta} = d\theta/dt = \vec{v}.\nabla\theta = v_\theta \vec{e}_\theta.\nabla\theta = v_\theta \partial\vec{x}/\partial\theta/|\partial\vec{x}/\partial\theta|.\nabla\theta = v_\theta = v_{//} sin\Theta/|\partial\vec{x}/\partial\theta|$; θ is the poloidal angle and Θ the angle between the total and the toroidal magnetic field. In absence of axisymmetry it is more logical to directly adopt the field aligned arclength variable $x_{//}$. Once departing from thermal equilibrium and going beyond Maxwellian distributions this will in any case become gradually more needed when the adopted model becomes more sophisticated. In that case the - in general non-Maxwellian - distribution function $F_o(\vec{\Lambda})$ appearing in the dielectric response term in the wave equation will need to be expressed in terms of $v_{//}$ and $v_\perp$ at each position in physical space while it is actually a function of the constants-of-motion variables $\vec{\Lambda}$ governing the trajectories imposed by the strong confining magnetic field in absence of the impact of the RF electric field (see e.g. [2]).

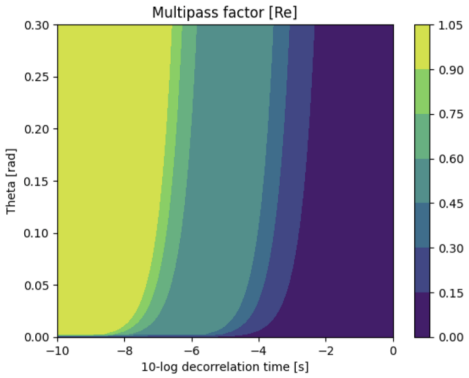

Figure 4. Multipass factor as a function of the (10-log) of the decorrelation time τ_d and the angle Θ between the parallel and the toroidal direction.

8 Conclusions

An overview was provided of a number of RF wave propagation and damping models maximally exploiting knowledge already available in the literature but combining them to enhance the model's realism. Most of the work presented in this paper is inspired on the desire to exploit numerical libraries such as NGsolve [12] allowing to sidestep needing to worry about numerics while being able to concentrate on the physics. The present paper is only a snapshot of ongoing efforts that will be carried further in the years to come.

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