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Implementation of a 2D wave-solver using the Budé method with NGSolve: initial validation without poloidal fields, with further investigation incorporating them

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Abstract. This paper presents the first 2D implementation of the so-called Budé approach to solve the integro-differential wave equation as a high-order differential equation. The approach has the advantage that it can readily be implemented in standard FEM codes and has a fit superior to the traditionally adopted Taylor expansion of the dielectric tensor. The method was originally proposed and tested in 1D for radio-frequency waves in absence of poloidal fields. Using the open-source finite element software NGSolve, the Budé method is reproduced in 1D for benchmark purposes and validated up to sixth order. The work is then extended to 2D models, demonstrating the ability to simulate wave propagation in realistic geometries as well as linear wave transformation processes, e.g. the fast wave confluence to the ion Bernstein mode. Although still missing a fully convincing model to capture poloidal effects, this paper therefore advocates the use of a new open-source numerical tool to concentrate on physics development, and doing so presents a first step towards efficient full-wave computations for wave heating fusion applications.

1 Introduction

In radio-frequency heating, finite temperature effects are crucial to determine the correct wave polarization and damping inside the plasma. However, including the finite temperature corrections (hence the Finite Larmor Radius (FLR)) to numerical models has long been an outstanding challenge in the fusion community. Two main ways of tackling this issue can be highlighted:

1. All orders codes, like AORSA [1], solve the all-FLR integro-differential wave equation in Fourier space with the continuous integral handled as a discrete sum. This method is accurate but is computationally heavy and too demanding to be routinely exploited in 3D simulations.

2. FLR approximation codes, like TORIC, EVE and TOMCAT [2–4], where the computational challenge is partly handled by using a truncated FLR approximation based on the assumption that the Larmor radius is much smaller than the perpendicular wavelength $\rho_L k_\perp \ll 1$. These codes are much faster, but break down under specific conditions. In the frequency domain of interest of this paper, the Ion Cyclotron Resonance Heating (IRCH), the fast wave launched by the antenna can mode convert after the resonance in an Ion Bernstein Wave (IBW) whose wavelength is comparable to the Larmor radius such that $\rho_L k_\perp \approx 1$.

A few years ago, Budé proposed a third approach profiting from the numerical efficiency of a truncated FLR approach and keeping the accuracy of an all-FLR code for the waves of interest *i.e.* $\rho_L k_\perp \approx 1$ in ICRH scenarios [5]. The method consists in creating a polynomial fit of the hot dielectric tensor

$$\epsilon_{ij}(\mathbf{x},\mathbf{k}) = \sum_{\substack{\mathbf{p} \\ p_x+p_y+p_z\le N_p}} c_{ij,\mathbf{p}}(\mathbf{x}) {k_x}^{p_x} {k_y}^{p_y} {k_z}^{p_z}, \qquad (1)$$

where N_p is the order of the polynomial and $\mathbf{p} = \left(p_x, p_y, p_z\right)$. The inverse Fourier transform property is then used to end up with simple differential operators in the wave equation. The hot plasma tensor is therefore approximated by a sum of derivatives:

$$\int_{-\infty}^{\infty} \sum_{j=1}^{3} \epsilon_{ij}(\mathbf{x},\mathbf{k}) \tilde{E}_j(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} \mathrm{d}\mathbf{k} = \sum_{\mathbf{p}} \sum_{j=1}^{3} (-i)^{p_x+p_y+p_z} c_{ij,\mathbf{p}}(\mathbf{x}) \partial_x^{p_x} \partial_y^{p_y} \partial_z^{p_z} E_j(\mathbf{x}) . \qquad (2)$$

This method was successfully demonstrated in 1D [5] and extended to arbitrary distribution in the absence of a poloidal magnetic field component [6]. A 2D semi-analytical analysis of the method, taking into account the poloidal field and arbitrary distributions, was also proposed and paved the way to port the method to 2D or 3D geometries [7]. The same paper also highlighted one

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possible weakness of the adopted procedure: the poloidal magnetic field component. Bringing the parallel dynamic into the fitting procedure, creates complications as it may require higher order polynomial fits, rendering the procedure impractical.

While the parallel dynamic may not be described correctly when adopting a too low order fit, some of the essential physics is likely to be captured nevertheless. With this mindset, the present paper applies the Budé method with a raw approach, aiming at demonstrating its potential for 2D simulations when using second order polynomial fit. The 3D full-wave simulation extension of the method will be demonstrated in a following paper as the method should be directly applicable but will likely face new unforeseen numerical challenges.

In this paper, a sixth order polynomial fit procedure of Budé in a 1D JET problem is first benchmarked with the original code outputs of Budé in section 2. To get to the Budé implementation in 2D, a staged approach was followed where we first implemented a cold FEMIC-inspired solver presented in section 3 and then ported the Budé method in 2D with a second order polynomial fit for a WEST and JET geometry without poloidal fields as presented in section 4. The challenges of the inclusion of the poloidal fields is then explored in section 5. Finally, some conclusion and prospects are discussed.

2 1D implementation

We first want to reproduce the work done by Budé in 1D. The work of Budé is described in [5] and the matlab code referenced therein used to reproduce the figures of his paper can be found on github and is open source [8].

The goal here is to reproduce a 1D case close to the one used in [5]. For this, the same JET case, using an analytical density and temperature profile, is used. The code can either excite a current or an electric field on the wall or at a specific place inside the 1D space of the machine.

To solve the wave equation, we decided to explore the potential of an open source FEM python software called NGSolve [9]. NGSolve is a high performance multiphysics finite element software with a convenient Python interface based on the weak formulation of the differential equation to be solved.

Here it is important to mention that the fitting procedure used by Budé in his original paper as well as the exact range of k_x on which the dielectric tensor is fitted is not the same. In our method, a least square polynomial fit was used for the fitting procedure. Furthermore, the numerical procedure rests on the FEM formulation whereas Budé discretized the high-order PDE with the finite difference scheme. Despite these facts, the agreement obtained between the two methods looking at the different field components is satisfying. This is presented in figure 1 where the absolute electric fields components obtained running the matlab code of Budé is compared with a sixth order fit performed with NGSolve. As it is a 1D code, it allows very fast computations (running time less than a minute on a 32 GB machine, intel core i9-10885H processor, smallest mesh size under the millimeter, and order 4 elements).

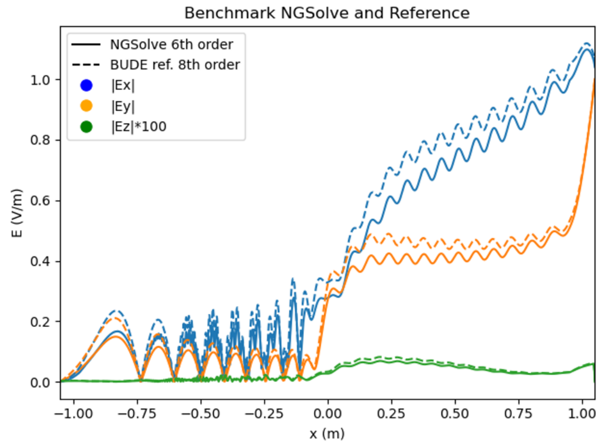

Figure 1. Benchmark of the JET case of Budé using the finite difference scheme with the NGSolve FEM case: comparison of the absolute electric fields components obtained with our FEM scheme and the one obatined with the finite difference scheme of Budé. Plasma parameters: analytical parabolic profiles, D-(H) plasma with a minority concentration of 5%, core density and temperature of $n_e = 7 \times 10^{19}$ m^{-3} and $T_e = 5$ keV.

3 2D FEMIC-inspired wave solver implementation

To get to 2D, a staged approach was adopted, where we first developed a basic 2D FEMIC-inspired code [10] with a cold or tepid dielectric tensor using NGsolve [9] before implementing the Budé method. This enabled us to familiarize with NGSolve features (boundaries and sources, elements order, etc.) and to verify key physical plasma wave aspects (wavelength, polarizations, resonance and confluences, etc.).

The FEMIC philosophy, described in [10] and recently extended to include aspects of the parallel non-local effects [11], solves the wave equation in a 2D axisymmetric geometry using a cylindrical coordinate system (R, Z), with a field decomposition of the form $\mathbf{E}(R, \phi, Z) = \mathbf{E}(R, Z)e^{in_{tor}\phi}$. The dielectric tensor $\mathbf{K}$ is computed for a given toroidal mode number n_{tor} and can include hot effects using a fixed $k_\perp = k_{fast}$ and assuming $k_\parallel$ as n_{tor}/R.

In 2D, the plasma equilibrium and static magnetic field components are reconstructed from eqdsk file using the open-source python software freegs [12]. The dielectric tensor can then be rotated $\mathbf{K} = \mathbf{R} \cdot \mathbf{K}^{Stix} \cdot \mathbf{R}$ from the plasma frame to the machine frame and can therefore include the confluence of the fast wave with the Ion Cyclotron Wave (ICW) while still missing the confluence with the Ion Bernstein Wave (IBW). Even though the poloidal magnetic field component is included in the calculations, its contribution is not considered in the $k_\parallel$. However, the method has a large spectrum of applications, and is quite accurate in most cases when the electron damping on IBW is not significant. Presently, the source is handled by imposing the fields at the antenna aperture. To enhance the poloidal and toroidal source description at the antenna aperture, the output electric field of the ANTITER code [13] given in Fourier space (k_y, k_z) is fourier inverted in

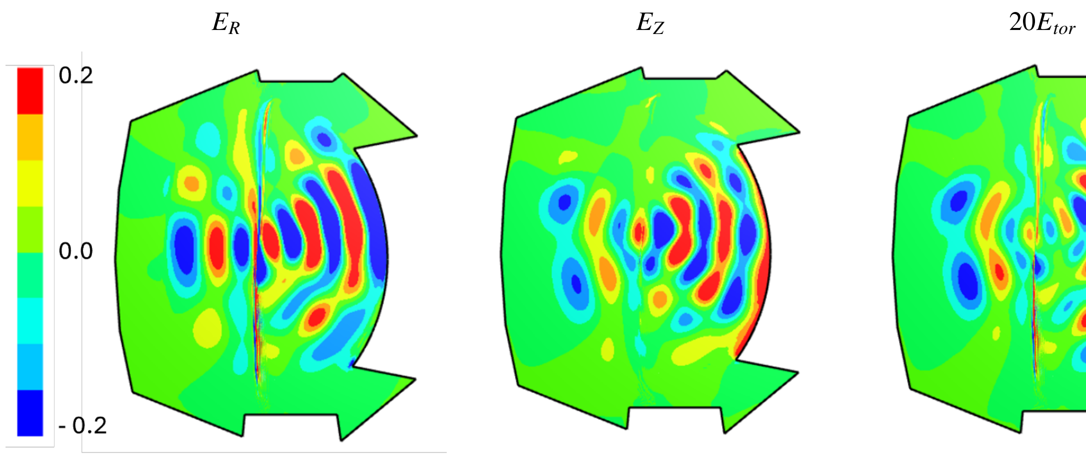

Figure 2. Output field components obtained for a WEST case using an external ANTITER excitation and for a given toroidal mode number $n_{tor} = 30$. Poloidal magnetic fields are included such that a clear conversion to the ICW is seen. Plasma parameters: D-(H) plasma with a minority concentration around 5%, core density and temperature of $n_e \approx 5 \times 10^{19}$ m^{-3} and $T_e = 1.2$ keV.

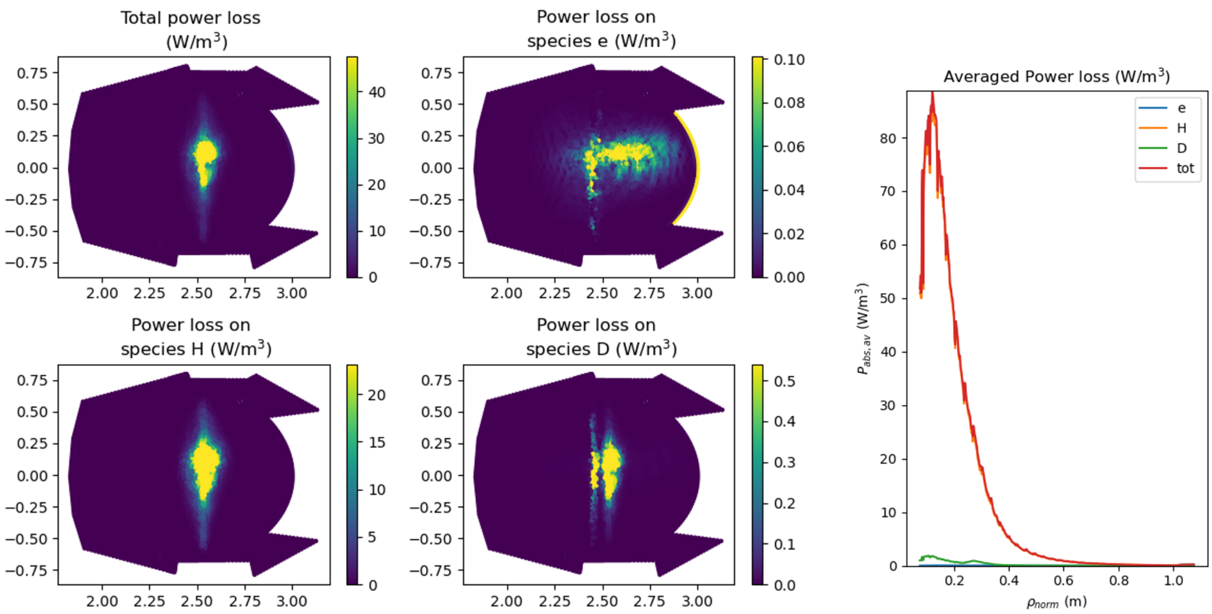

Figure 3. Post-processing of the power loss per species in 2D (left) and averaged on flux surfaces (right) for an ANTITER dipole excitation of 1A of a flat Q2 antenna.

the poloidal direction k_y and can then be imposed to the input source of NGSolve.

The output fields obtained for a given toroidal mode number $n_{tor} = 30$ is presented in figure 2. The FEMIC approach also enables to reconstruct the total electric field excited by the antenna at a given toroidal angle ϕ. It is therefore possible to perform a python post-processing to extract the fields and compute power losses for different plasma species as illustrated in figure 3. The approach also allows to reconstruct a 3D field pattern although not presented here.

4 2D Budé implementation without poloidal fields

In this section we demonstrate the Budé method, building upon the FEMIC-like tool created in our previous section.

The same WEST case was taken except that the poloidal static magnetic field component was removed. The confluence with the IBW is thus introduced while the ICW confluence is excluded. The dielectric tensor is fitted with a second-order polynomial fit leading to second-order differential operators in RZ.

This case, presented in figure 4, has a clear signature of the ion Bernstein wave, seen in the E_R and E_{tor} components but not on the fast E_Z component. NGSolve also has a built-in movie capability, allowing to see the electrostatic nature of that wave, superimposed on the propa-

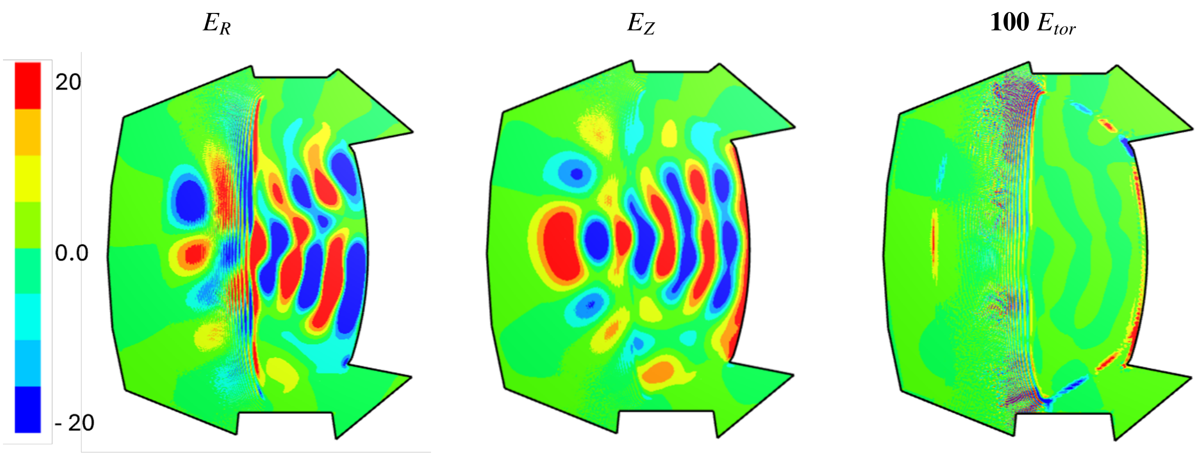

Figure 4. Output field components obtained for a WEST case for a given toroidal mode number $n_{tor} = 30$ and a constant $E_y = 20$ V excitation at the antenna using the Budé method with a second order polynomial fit. Poloidal magnetic fields are not included such that a clear conversion to the Bernstein wave is observed.

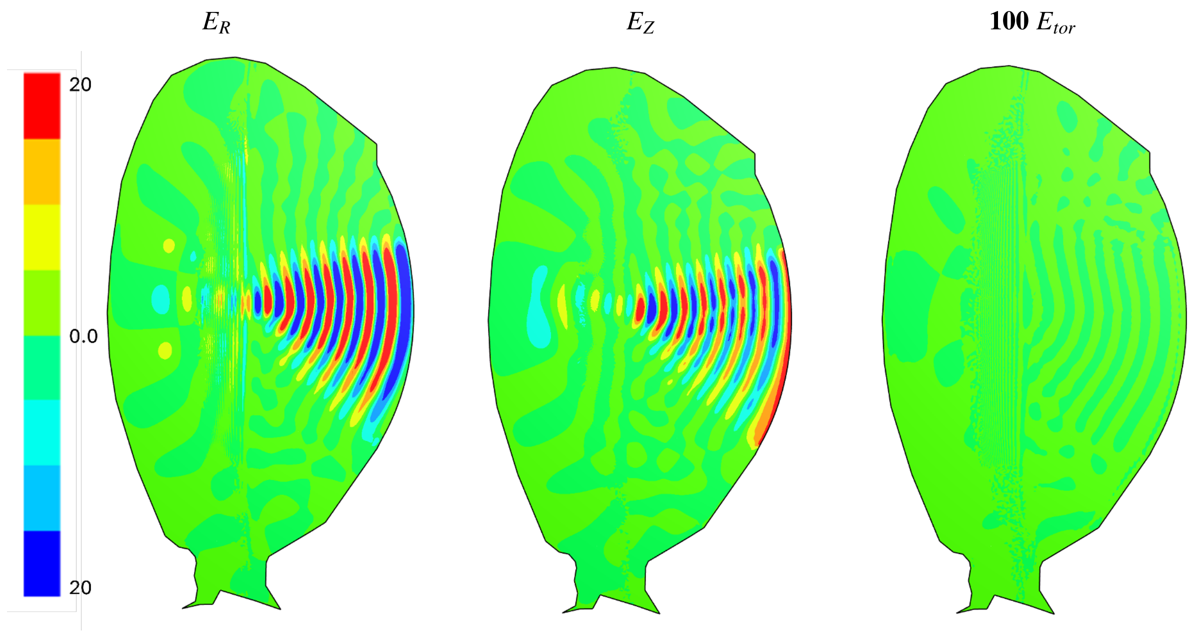

Figure 5. Output field components obtained for a JET case for a given toroidal mode number $n_{tor} = 27$ and a constant $E_y = 20$ V excitation at the antenna using the Budé method with a second order polynomial fit. Poloidal magnetic fields are not included such that a clear conversion to the Bernstein wave is observed.

gating fast wave field pattern. As the toroidal electric field is aligned with the magnetic field in this case, the toroidal component is very weak and enhancing it gives the possibility to see the resolution limit of the simulation (running time below 5 minutes on a 32 GB machine, intel core i9-10885H processor, smallest mesh size 1 cm, and order 3 elements). For cases in absence of poloidal magnetic field components, the wave plasma response seems relatively independent of the $k_\perp$ range used for the fitting procedure, giving us confidence in the global wave patterns presented.

The same exercise can be performed with JET as presented in figure 5. In the case of JET, the fast wave is localized when reaching the core of the plasma and the IBW wave signature is therefore rather discrete and only seen in the core.

5 2D Budé implementation with poloidal fields

While the results presented are already a huge step towards accurate full-wave computations, it is still lacking the essential ingredient of the poloidal magnetic field which will bring the extra confluence with the ICW [14, 15].

As mentioned earlier, retaining the poloidal field component brings additional difficulties in the accuracy of the dielectric tensor polynomial fitting. Several problems can arise in this specific case:

- With the current approach, the positive definite power balance is not guaranteed. This can create artificial modes that are not expected in reality. This could be cured using Kennel-Engelmann operators as described in a companion paper of the conference [16].

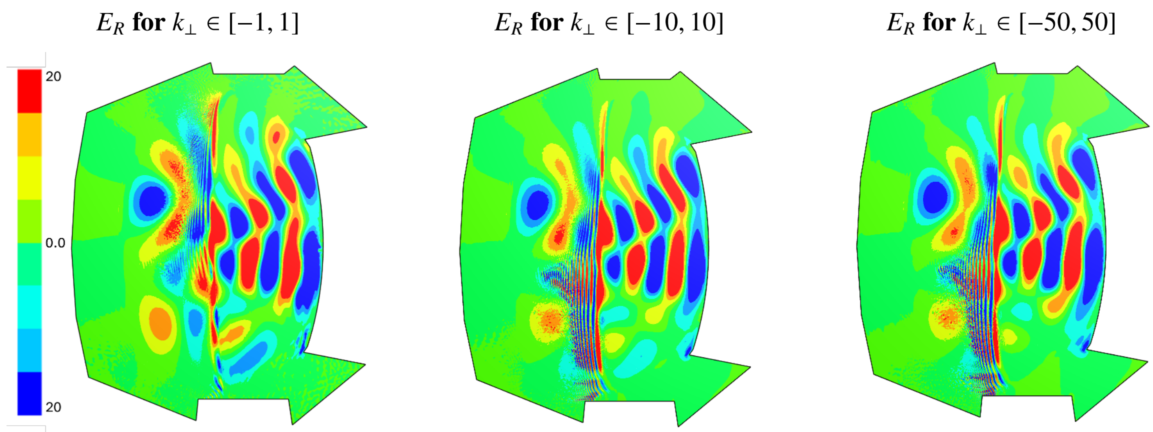

Figure 6. Output field components obtained for a WEST case for a given toroidal mode number $n_{tor} = 30$ and a constant $E_y = 20$ V excitation at the antenna using the Budé method with a second order polynomial fit over different $k_\perp$ range. Poloidal magnetic fields are included such that a competition between IBW and ICW is visible.

- High poloidal magnetic field components, such as those found in the case of JET, lead to poor fitting of the dielectric tensor, which leads to non-physical wave dynamic e.g. unexpected cut-offs.

These problems could most likely be cured with higher order polynomial fits. However, this itself leads to higher order differential operators, which creates new numerical challenges that are not yet completely solved with NGSolve.

One can nevertheless find successful cases incorporating poloidal fields. In such cases however, the plasma wave response depends on the adopted $k_\perp$ range used to make the fit. An example is given for WEST (small poloidal angle and no parasitic mode excitation) in figure 6 for three different fit range of $k_\perp$. Adopting a rather small $|k_\perp|_{max}$ should mimic aspects of the traditional truncated FLR while adopting a larger range gradually adds supplementary aspects. At this moment, the exploitation of NGsolve on poloidal cases is in development, so all findings will need to be further examined and understood.

6 Conclusion

This paper briefly presents the first implementation of the Budé approach, using higher-order differential operators to include key FLR effects such as the ion Bernstein wave. The method works smoothly in the case of no or weak poloidal magnetic field components, such that found in a tokamak like WEST, and represents an important step towards describing the fate of RF waves in hot plasmas more realistically: it supersedes fast-wave-only models and provides an improved way to assess the dynamic in the mode confluence region. However, the method still faces challenges in the case of higher poloidal magnetic field components conditions, such as those found in a tokamak like JET. On the other hand, this work demonstrates the potential of FEM software packages such as NGsolve to concentrate on physics development.

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