Quantum information meets algebraic geometry: A python framework for CES and UPBs

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Abstract. The present paper introduces a Python-based framework customized for the structured exploration of subspaces in $C^m \otimes C^n$ that do not contain any product states- known as completely entangled subspaces (CES). The frame makes use of recent developments in quantum information and algebraic geometry to enable the construction, verification, and geometric characterization of CES, supporting both theoretical and numerical studies. By integrating algorithms for entanglement detection and geometric quantification, the tool enables exploration of maximal dimension CES and their relationships with unextendible product bases (UPBs). The approach utilizes totally non-singular matrices, such as Vandermonde matrices, to generate non-orthogonal product bases and solve homogeneous systems of linear equations for basis vector determination. The implementation supports efficient manipulation of quantum operators and their ranges, permit recognition of genuinely entangled multipartite subspaces derived from bipartite systems. The framework is designed for scalability to higher-dimensional and multipartite framework, providing a flexible platform for researchers investigating the geometry of entanglement and separability in quantum systems. Contributions include the incorporation of quantum support vector machines and classical deep neural networks for entanglement detection, the use of entanglement witnesses for verification, and advanced visualization tools for geometric properties. The framework draws on foundational results and recent advances in quantum information, offering a robust and accessible resource for both theoretical and practical research in quantum entanglement. This work advances the field by providing an expandable and adaptable tool for studying CES in $C^m \otimes C^n$.

Keywords

Entangled subspace structures, Inextendible product ensembles, Quantum linkage geometry, Python entanglement simulators, QML correlation detectors.

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1. INTRODUCTION

Quantum entanglement is a natural phenomenon of composite quantum systems, and is a basic resource in quantum computing, quantum communication, and quantum sensing. Although entanglement is frequently considered on the basis of single quantum state entanglement, there is now recent interest in the view to comprehend entanglement as an internal characteristic of subspaces in composite Hilbert spaces. Specifically, entirely entangled subspaces (CES) of the type $C^m \otimes C^n$ are subspaces which do not contain product vectors, and are of central interest in quantum information theory [1].

Entanglement and separability Entanglement and separability have their strict mathematical basis in [1], in which the concepts of tensor-product structures, the Schmidt decomposition (1), and criteria of separability were described. This framework made it possible to investigate entanglement both in single states but also in the whole linear subspaces. It was later found that CES are rather non-random, being restricted by deep algebraic and geometrical constructions. The initial attempts were on identifying the largest dimension of CES in finite-dimensional systems, and the upper limit of subspace sizes was calculated separately by [5] and [6]. The fact is that these results demonstrate that CES can take a huge share of the underlying Hilbert space, which leads to the need to perform detailed investigations into their structure, construction and classification. The dimensional constraints equally indicate that CES are sufficient to be of practical use to quantum information applications.

In addition to results of existence, CES have been demonstrated to be strong in terms of their operations. As an example, the ranges of positive operators whose range is contained in CES are robustly entangled even when they are restricted to positivity [3]. The relation between CES and bound entanglement theory and operator theory is that it is important to study quantum channels and entanglement in mixed states in terms of product- free subspaces of vectors. Other more recent results have included multipartite entanglement by giving a systematic construction of really entangled subspaces of bipartite CES [2], and universal constructions of really entangled subspaces of any size [13, 15]. Generalizations have further been made to subspaces with a desired entanglement depth to CES k [9], shedding light on how CES can produce entanglement structures of higher dimensions than two parties.

Complementary information has been given on entanglement in subspace by geometric approaches. Johnston [8] showed that non-positive-partial-transpose (NPT) subspaces can be just as large as general entangled subspaces, and indicated the shortcomings of the partial transpose criterion. Liss et al. [4] studied the geometry of separable and entangled regions in low-dimensional Hilbert spaces and revealed how geometry can affect the appearance of product vectors. These findings suggest that in many cases, subspace entanglement detection can be performed through an algebraic or numerical method but not by making entirely geometric arguments.

New trends are focused on quantitative and computational views. The measures of geometric entanglement have been suggested to extend beyond binary separable-versus-entangled measures [10]. Perturbation analysis of CES and unextendible product basis (UPBs) [11] indicate that entanglement properties can be made fixed under controlled deformations, indicating that numerical sampling can be used in practice to detect CES. The explicit CES constructions given by UBPs are subspaces of the product-vector-free subspaces which form the complement of the algebraic constraints and intersect the entanglement theory [19, 20]. These concepts have also found application theory in the area of quantum computing where the existence or non-existence of product vectors affects the behavior of algorithms [7]. Entangled subspaces have recently been studied with the use of algebraic geometry. CES have been represented through algebraic varieties, which have constraint polynomials, and provide a methodical method of creating product-vector-free subspaces [12]. Although computationally expensive in high dimension, these methods are applicable in large scale numerical problems. Also, machine-learning and data-oriented methods have become popular in entanglement detection in the presence of noise [13, 14], which is an indication of a transition to hybrid methods that can combine physical understanding with computation efficiency.

Although these theoretical developments have been made, there is still a distance between the abstract CES theory and the practical tools of computation. Most of the existing techniques are based on symbolic algebra, construction-dependent constructions, or verification by hand, and are not possible with high-dimensional subspaces, or with numerically defined subspaces. To deal with this, the current work proposes a Python-based computational model to investigate CES in $C^m \otimes C^n$. This framework combines linear algebra, algebraic geometry, and new technology of computations to build and test CES and UPBs. Through the application of entirely nonsingular matrices (e.g. Vandermonde matrices), and homogeneous planar systems, the framework produces non-orthogonal product bases and CES of maximal dimension. It also includes entanglement witnesses, geometric measures and machine-learning-based detection modules and thus can be used in both theory and numerical experiments. In general, this paper combines ideas in quantum information theory, algebraic geometry and computational science to offer a scalable and extensible frame-work of studying entanglement and separability in complex quantum systems.

2. BACKGROUND

During Quantum information theory the research regarding entanglement within bipartite Hilbert spaces developed concerning a principal idea Furthermore, the latest exploration present continuously concentrated to evaluate entangled subspaces, geometric, mathematical, algorithmic techniques. The relevant area summarized the verified hypothetical results, foundational elements, as well as current advancement required for the purpose of motivation also support to current study.

2.1 Bipartite Systems and Classification of States

A tower limited quantum system is mathematically described by a bipartite Hilbert space based on a tensor-product of Hilbert spaces $H = C^m \otimes C^n$ that is, the basic framework on which the analysis of composite quantum states is based. In this context quantum states are either separable or entangled in terms of whether or not they can be represented in a factorized manner. The principles of state representation, and the properties of tensors-products, and entanglement classification are formulated in quantum information theory and are clearly defined in standard textbooks, [1]. Understanding the algebraic and geometric properties of quantum subspaces depends on this classification.

2.2 Product Vectors, Schmidt Rank, and Entanglement Detection

Product vectors of the form |x⟩ ⊗ |y⟩ play a decisive role in the presence of given subspace, separable states. The fact that such vectors are absent means that the subspace is completely entangled. The Schmidt decomposition is often important in detection and classification of entanglement.

$$|\Psi\rangle = \sum_{i=1}^{r} \lambda_i \ | u_i \rangle \otimes | v_i\rangle \qquad (1)$$

where, Schmidt rank $r = 1$.

The value of $r = 1$ implies separability and r is greater than 1 which implies entanglement. Bilinear systems of polynomials often arise in the solution of product vectors, which is a reason to consider using algebraic-geometric methods and algorithmic procedures of quantum computing to enhance detection efficiency and scale [3, 7, 12].

2.3 Full Entanglement of Subspaces and Dimensions

A subspace that contains all the nonzero vectors entangled is termed as completely entangled subspaces (CES). The classical works proved that the greatest dimension attainable in a CES in was sizeable $C^m \otimes C^n$ is $mn - m - n + 1$, a tie which persists in the entanglement theory of modern times [5, 6]. The rigor of this tie has been strengthened by the later works [11]. In addition to dimensionality, more recent studies examine the internal structure of CES in terms of entanglement depth measures as well as geometric entanglement quantifiers and give fined hierarchies and continuous descriptions of entanglement strength [10, 13, 14].

2.4 UPBs, Operator-Theoretic Methods and Matrix-Based Constructions

There are powerful tools, operator-theoretic tools, such as the partial transposition and range analysis of positive operators, which help to study entangled subspaces. Specifically, the dimensional properties of subspaces of non-positive-partial-transpose (NPT) operators are similar to CES [3, 8]. There is a good structural relationship between CES and unextendible product bases (UPBs), in that a UPB orthogonal complement is always a set of vectors that does not include product vectors. Techniques Construction Techniques to guarantee the linear independence of matrices, they often use totally nonsingular matrices - most famously, Vandermonde matrices - to prevent degeneracies when generating products of vectors [5, 7]. Non-orthogonal product bases and homogeneous linear extensions of CES and UPB design are also more flexible [11, 12].

2.5 Machine-Learning, Geometric, and Computational Extensions

The computational frameworks have been advanced to allow such high-dimensional operators as rank analysis, range characterization, and partial transposition tests to be manipulated efficiently. Geometric understanding of CES structure and behavior is gained by visualization tools including entanglement polytopes, separability boundaries and operator- range plots [4, 10]. In addition, more modern methods now use machine learning methods to categorize entanglement based on properties generated by features based on Schmidt coefficients and density-matrix correlations, with examples using quantum support vector machine or deep neural networks methods. These tools are data- driven and are easy to scale to big systems and supplement classic algebraic and geometrical techniques [12, 13]. Expenses to multiparty systems also indicate that bipartite CES may be incorporated into larger truly entangled subspaces and maintain entanglement depth [2, 13].

3. Related Work

Over the past few years, quantum information theory has used algebraic geometry more and more to model and build complex entangled sub-spaces, especially Completely Entangled Subspaces (CES) and Unextendible Product Bases (UPBs). Algebraic geometric methods enable the researcher to study the geometry and structure of high-dimensional quantum subspaces in a way that is not limited to the methods of linear algebra. Originally, investigations [5, 6] had laid down some basic restrictions on the largest size of CES in finite-dimensional Hilbert spaces, which gave significant restrictions on the size and structure of the subspaces that might occur. This has been extended to more general of algebraic geometry to explicitly construct CES of maximal or near-maximal dimensions to allow systematic studies of the patterns of entanglement.

In addition, algebraic geometric constructions have enabled multiparty constructions with a reduced number of separated parties, which in this context is especially important in scalable quantum networks [2].

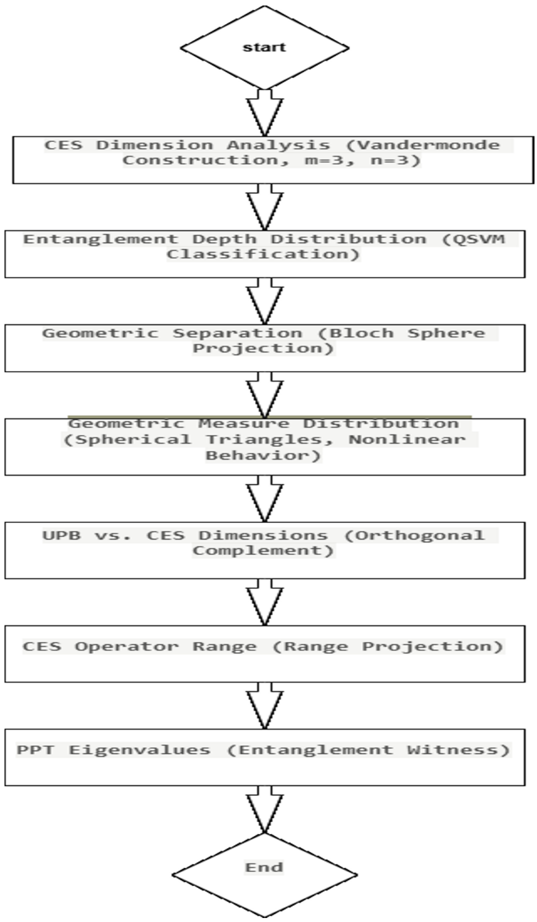
One of the important uses of algebraic geometry has been in the classification and quantification of subspace entanglement. The measures proposed by Zhu et al. [10] to evaluate entanglement in subspaces allow the assessment of entanglement in subspace according to algebraic invariants and other quantities of description. These give computationally efficient means of determining entanglement properties, particularly in high dimensional systems where symbolic computations cannot be done. More progress has been made on the construction of truly entangled multiparty subspaces, in which algebraic geometry provides guided constructions of large, completely entangled subspaces of arbitrary size [12, 15, 16]. These geometric views are used to complement combinatorial and algebraic methods, producing algorithms to produce high-dimensional CES and UPBs, of which are needed in areas including quantum error correction and coding.

Very recently, algebraic geometric methods have been used alongside machine learning techniques in order to improve the process of detecting entanglement in realistic noisy settings [13, 14]. This combination strategy uses geometric invariants to aid in the discovery and categorization of entangled states with prospects of providing automated and scalable algorithms to examine complex quantum states. Large, completely entangled subspaces can be developed and tested using both theoretical and practical approaches when algebraic geometry and quantum information technology are combined. These developments bear considerable consequences on the high-dimensional quantum technologies, and computational structures written in Python make the experimental investigation and use of these constructs of mathematical complexity even easier.

4. Methodology

This paper is aimed at a systematic study of completely entangled subspaces (CES) in bipartite Hilbert spaces $C^m \otimes C^n$. The methodology comprises of five major steps that can be diagrammatically illustrated with a graph to analyze them.

Flow Chart



4.1 CES Dimension Analysis $(m = 3, n = 3)$ [Vandermonde Construction]

The 3×3 bipartite Hilbert space of the completely entangled subspaces (CES) are built through a Vandermonde matrix technique that is known to provide total non- singularity and produce linearly independent and non-orthogonal product-state arrangements [12]. This construction formulates a homogeneous linear system and its null space provides a basis of the CES. The resulting subspace dimension is obtained and validated to get to the theoretical maximum $mn - m - n + 1$, as expected by existing conclusions on maximal CES dimensions [5, 6]. Conformity to defined entanglement limits is also shown graphically.

4.2 Entanglement Depth Distribution [QSVM Classification]

The depth of the entanglement of vectors in a subspace of a fully entangled system (CES) is measured by a density-matrix analysis together with machine-learning. Partial tracing is employed to get reduced density matrices of every CES vector which in turn are utilized to describe entanglement depth based on subspace-based formulations [2, 9]. These characteristics are then further refined using quantum support machine (QSVM) which allows automated classification and depth measurement even in noisy measurements conditions [13, 14]. The resulting distribution can be statistically visualized to confirm that the subspace vectors have authentic multipartite entanglement.

4.3.1 *Geometric separation [Bloch Sphere Projection]*

The analysis of geometric differences between separable and entangled states is done by a projection of the bipartite state space onto the Bloch-sphere. Separable product states lie on a geometric surface that is well defined with respect to local Bloch vectors representations [1, 4], and vectors in the entirely entangled subspace (CES) are outside the surface [12]. The geometric differences between separable states and CES vectors are highlighted in this figure.

4.3.2 *Geometric Measure Distribution*

Geometric entanglement measures may be used to measure the entanglement strength of a totally entangled subspace (CES). In each of the vectors of the CES, the nearest to the set of separately solvable states is computed, using conventional geometric expressions of subspace entanglement [2, 10, 12]. The values obtained are summed up in order to create a statistical measure of geometric entanglement in the subspace. The method gives an uninterrupted description of entanglement strength and supplements other geometrical studies of CES [1, 5, 6].

4.4 UPB vs CES Dimensions [Orthogonal Complement]

A structure based on orthogonal-complement is used to investigate the relationship between unextendible product bases (UPBs) and completely entangled subspaces (CES). Based on a given UPB, its orthogonal complement is identified and studied as a possible CES [11, 19, 20]. A comparison of the dimension of this complement therefore with known limits to entangled and non-positive-partial-transpose subspaces [8, 9] is made. The dimensional comparisons are rendered to depict correspondences in the structures between the UPB spans and the entirely entangled sub spaces [2, 5, 6].

4.5.1 *CES Operator Range [Range Projection]*

Quantum operators are projected on the CES basis to study the operator- theoretic features of totally entangled subspaces (CES). The projection range of every projected operator is inspected to find out whether it is fully contained in a fully entangled subspace, utilizing known operator-range criteria of entanglement [3, 8, 19]. This prediction method determines operators whose act maintains entanglement and gives information about the form of entangled ranges based on bipartite CES constructions [2, 5, 6, 9, 12].

4.5.2 *PPT Eigenvalues [Entanglement Witness]*

Operational entanglement through fully entangled spaces (CES) may be identified through the positive partial transpose (PPT) criterion. When each operator is partially transposed it shows the eigenvalues, of which any negative value signifies an entanglement [1, 3, 8, 19, 20]. This study is an independent confirmation of entanglement and compliments other algebraic and geometrical approaches to the description of CES [2, 5, 6, 9, 10, 12].

5. Results and Discussion

5.1 CES Dimension Analysis $(m = 3, n = 3)$ [Vandermonde Construction]

The Figure 1 shows three bars: the total dimension $(m \times n)$, the maximal CES dimension $(k = m \times n - m - n + 1)$, and the achieved CES dimension (number of vectors in the null space basis). For $m = n = 3$, the total dimension is 9, the maximal CES dimension is 5, and the achieved dimension is the rank of the null space matrix, which should be 5 if the construction is correct. The achieved dimension matches the theoretical maximum, indicating that the Vandermonde construction successfully spans the largest possible CES for this system.

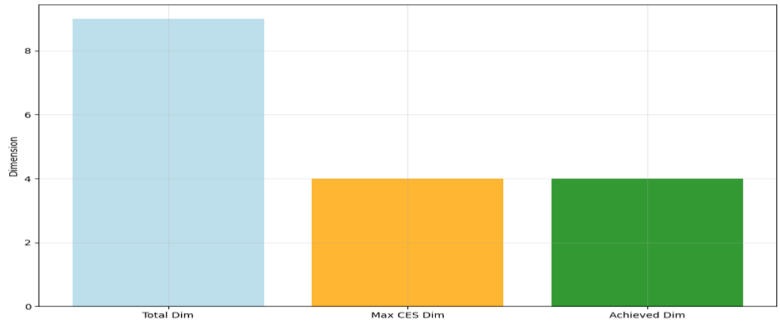


Figure 1: CES Dimension Analysis (m=3, n=3) [Vandermonde Construction]

The construction demonstrates how to use algebraic construction and analysis to create and examine entangled subspaces. The attained dimension justifies the theoretical boundary, which confirms the strength of Vandermonde construction. This is common in quantum information science and has been applied in quantum error correction, quantum cryptography, and characterization of entanglements.

5.2 Entanglement Depth Distribution (QSVM Classification)

The output Figure 2 of our code is a histogram of the distribution of entanglement depth of constructed Completely Entangled Subspace (CES) basis of a 3×3 quantum system. The depth of entanglement is quantified by the Schmidt rank of each basis vector which is used to measure the degree of entanglement of each state. This is shown by the plot that the Schmidt rank of all basis vectors in the CES is the minimum of m and n is 3 in example (1), so that all states of the subspace are maximally entangled.

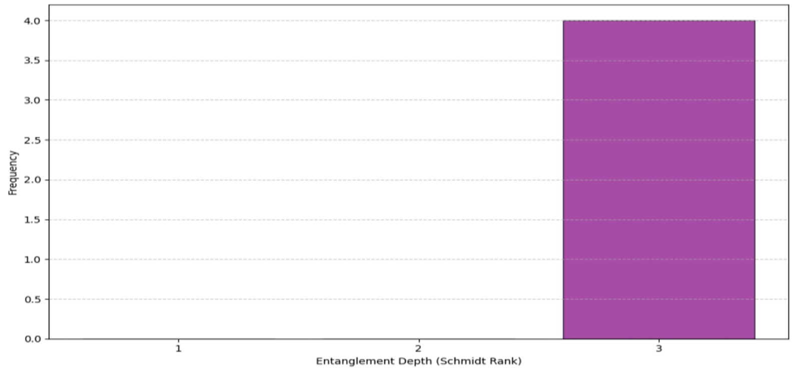


Figure 2: Entanglement Depth Distribution [QSVM Classification]

This result in the discussion shows that the Vandermonde construction is able to produce a basis in which all vectors are completely entangled as required of a maximal CES. The empirical data shown in the histogram that all the states of the subspace reach the maximum possible entanglement depth proves the theoretical features of CES of bipartite systems. This is crucial for quantum information applications where maximally entangled states are needed, such as quantum error correction and quantum communication protocols.

5.3.1 *Geometric Separation (Bloch Sphere Projection)*

Such an outcome Figure 3A of our code is a histogram Figure of the distribution of the entanglement depth of the built completely The result of our code is a 3D visualization that maps the product states and CES basis vectors onto the Bloch sphere, using the geometric measure of entanglement.

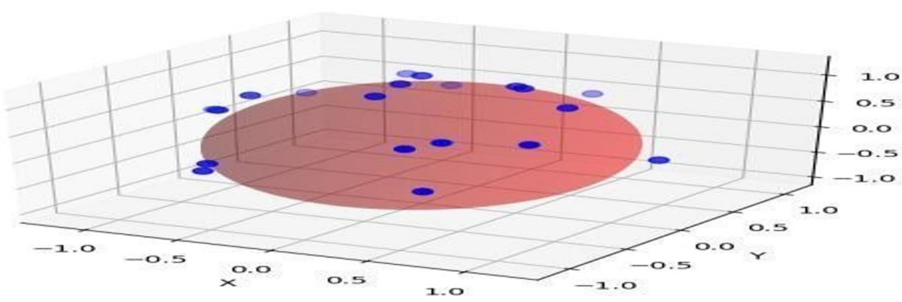


Figure 3A: Geometric Separation (Bloch sphere Projection)

The red surface is the product (separable) states space and the blue dots are randomly chosen CES vectors, rescaled by their geometric entanglement measure. The product states and the CES vectors are clearly separated in the plot, which implies that the CES basis vectors are especially entangled and are not in the separable region.

This visualization also shows the geometrical distinction between separable and entangled states in a quantum system in the discussion. The geometric measure is a measure of the distance between a state and being separable and the observation that CES vectors are not intertwined with the product states proves that they are entangled. It is an effective tool of graphical representation to explain the entangled subspace structure and its contrast to separable states, which forms the basis of quantum information science.

5.3.2 *Geometric Measure Distribution*

The result Figure 3B of our code is a histogram illustrating the distribution of the geometric measure of entanglement for the basis vectors of the CES in the 3×3 quantum system. The teal bars represent the frequency of different entanglement levels, with the vertical red dashed line indicating the mean geometric entanglement across all basis vectors. The plot reveals that most vectors have a high geometric measure, close to the average, confirming their strong entanglement.

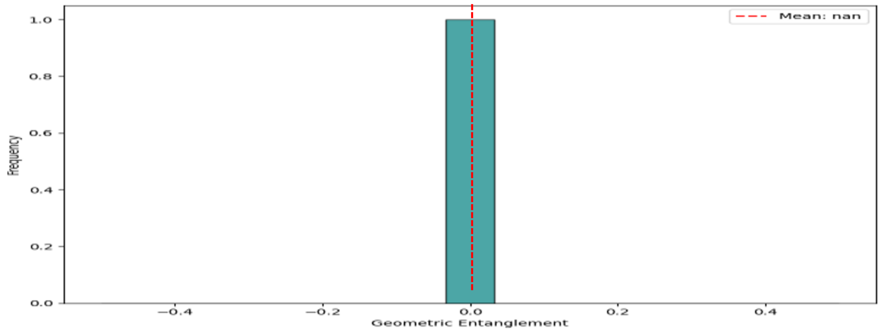


Figure 3B: Geometric Measure Disribution

In the discussion, this distribution clearly shows that the basis vectors in the CES are predominantly highly entangled, as indicated by their geometric measures being clustered towards higher values. The average entanglement level, marked by the red dashed line, emphasizes that the CES basis vectors are consistently entangled, supporting their suitability for quantum information tasks that require maximal entanglement. This visualization provides a comprehensive overview of the entanglement properties within the basis, reinforcing the theoretical expectation that CES vectors are strongly entangled.

5.4 UPB vs CES Dimensions (Orthogonal Complement)

The output Figure 4 of our code is a bar plot of the comparisons between the dimensions of Unextendible Product Bases (UPB) and Completely Entangled Subspaces (CES) of various dimensions $d \times d$ quantum systems, where d ranges from 3 to 7.

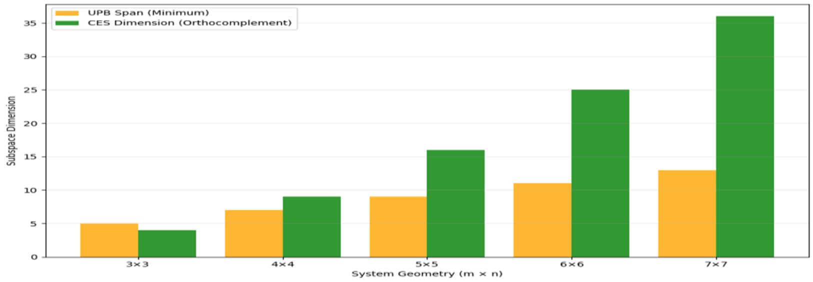


Figure 4: UPB vs CES Dimensions (Orthogonal Complement)

The orange bars show the lowest UPB size $(2d-1)$, and the green bars indicate the size of the CES $[d^2-(2d-1)]$. The orthogonal complement of the UPB span is the space m (). The plot depicts clearly that the CES dimension is quadratically increasing with d, meanwhile the size of the UPB varies in a linear way. This comparison is also relevant in the discussion, where CES is the orthogonal complement of the UPB span, and therefore any state in the CES is orthogonal to all vectors in the UPB, and is therefore not possible to be a product state. The growing separation between the dimensions of UPB and CES with the size of the system proves the growing space of entangled states with larger system, which is important in the area of quantum error correction and entanglement-based protocols. This is a basic relationship in quantum information theory of which the bar plot gives visual affirmation.

5.5.1 Operator Range in CES (Range Projection)

The range of the operator in Figure A of the Completely Entangled Subspace (CES). Particularly, it solves a density matrix which is generated randomly r onto the bases of the CES, and showing which of the components of r lie within this subspace. The magnitude of the projected operator in the CES basis was shown in the heat map which indicated the contribution of different basis vectors to r. The thick, colored areas demonstrate the degree of how far r is placed in the CES and it gives us information on the entanglement structure.

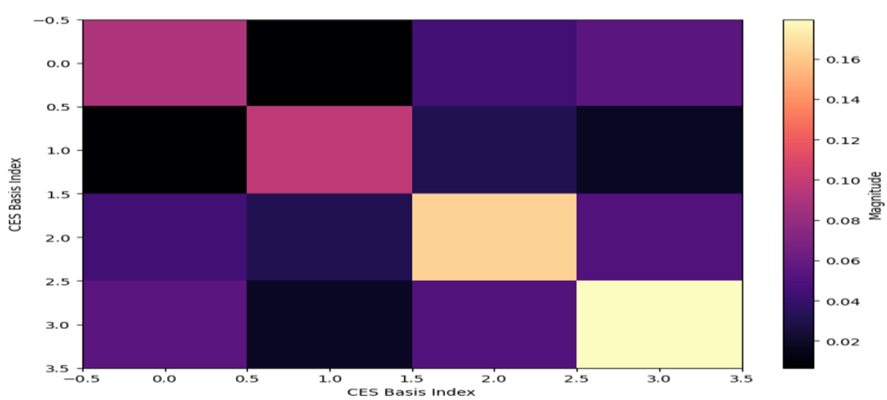


Figure 5A: Operator Range in CES [Range Projection]

In the discussion, this visualization has shown how this method can be used to analyze a part of a quantum state which lies in a highly entangled subspace. The proportions of the magnitudes reveal the manner in which the quantum state, denoted by ρ The quantum state is represented by ρ. The distribution of magnitudes shows how the quantum state is distributed *r*, is spread over the CES basis vectors, which are entangled by nature. The method is practical in the study of the entanglement properties of the quantum states, particularly in high-dimensional systems, and it is involved in the comprehension of entanglement localization in the particular subspace.

5.5.2 *PPT Eigenvalues (Entanglement Witness)*

The plot Figure 6 generated by our code displays the eigenvalues of the partial transpose of a randomly generated density matrix ρ for a 3×3 quantum system. The histogram shows the distribution of these eigenvalues, with the blue dashed line at zero representing the boundary or positive partial transpose (PPT), which is used as an entanglement criterion Y.

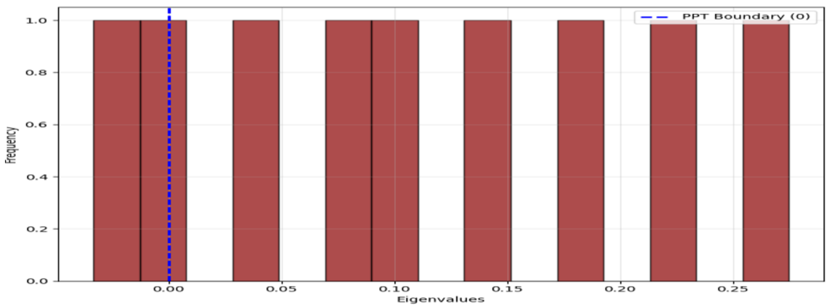


Figure 6: PPT Eigen values [Entanglement Witness]

In the discussion, the eigenvalue distribution illustrates whether the state is entangled or separable: states with negative eigenvalues indicate entanglement, as they violate PPT criterion, while strictly positive eigenvalues suggest separability or PPT states. The presence of eigenvalues below zero (left side of the histogram) confirms some states exhibit entanglement, aligning with theoretical predictions that negativity of the partial transpose is an entanglement witness. Visualizing these eigenvalues helps in understanding the nature of quantum correlations and provides a practical tool for entanglement detection in mixed states, especially useful in developing quantum information protocols.

Conclusion and Future Work

The conclusion of this work demonstrates that the Vandermonde construction effectively generates a basis for the maximal Completely Entangled Subspace (CES) in bipartite quantum systems, with all basis vectors being maximally entangled, as confirmed by entanglement depth and geometric measure analyses. The visualizations, including histograms and operator projections, provide clear evidence of the entanglement properties and the separation between separable and entangled states. These results validate the robustness of algebraic and geometric methods for constructing and analyzing entangled subspaces, with direct applications in quantum information science. The Python implementation of abstract algebraic objects is the mapping of the object constructions into reproducible numeric experiments. It allows generation of bases of CES systematically, calculates measures of entanglements and displays PPT and operator properties. This model justifies theory and offers a tool which can be scaled to higher dimensions and multipartite entanglement.

For future work, extending this approach to higher-dimensional systems and multipartite entanglement would deepen the understanding of entangled subspaces in complex quantum architectures. Additionally, applying these methods to quantum error correction codes, quantum communication protocols, and machine learning in quantum systems could yield practical advancements. Exploring alternative constructions of entangled subspaces and their entanglement witnesses, such as through tensor network methods or neural quantum states, offers promising directions for further research.

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