

# Hyers-Ulam (HU) Stability and Mittag-Leffler-HU Stability of Additive Brownian Motion (Louis Bachelier's Model)

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**Abstract.** The concept of Hyers–Ulam stability provides a powerful framework for analysing the robustness of functional and differential equations (DE's) under small perturbations. In this study, we investigate the Hyers–Ulam stability and Mittag-Leffler–Hyers–Ulam stability of Louis Bachelier's stochastic DE's, identifying conditions under which approximate solutions remain close to exact ones. By extending the classical stability concept, we establish sufficient criteria ensuring that if a function approximately satisfies the given stochastic equation, then a true solution exists in its neighbourhood. The analysis emphasizes the relationship between bounded deviations and the structural properties of the equation, offering insight into the qualitative behaviour of dynamic systems influenced by uncertainty. These results enrich stability theory and have applications in stochastic processes, control theory, numerical analysis, and mathematical modelling. Furthermore, we examine the relevance of the Bachelier's model to carbon control modelling. The findings show that the system exhibits stability over finite time intervals, meaning small perturbations produce proportionally small deviations. This supports reliable short-term forecasting and policy decisions in carbon pricing. However, due to the unbounded variance of Brownian motion, stability does not persist over long horizons, limiting its effectiveness for long-term environmental planning.

## 1 Hyers-Ulam (HU) Stability

The concept of stability plays a fundamental role in the study of differential equations, particularly in understanding how small perturbations in initial conditions or functional forms affect the behaviour of solutions. One important type of stability, known as Hyers-Ulam stability, originates from a problem posed by Ulam in 1940 [21] concerning the stability of functional equations. Ulam, in a lecture at the University of Wisconsin, asked whether a functional equation that is “approximately” satisfied necessarily implies the existence of an

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exact solution close to it. This question marked the beginning of a new perspective in the theory of functional equations -- one focused on stability under perturbations.

In response, Donald H. Hyers (1941) [11] provided the first rigorous answer to Ulam's question in the setting of Banach spaces. He proved that “if a mapping  $F: X \rightarrow Y$  approximately satisfies the Cauchy functional equation  $f(x + y) = f(x) + f(y)$ , then there exists a unique additive function  $T$  such that the deviation between  $f$  and  $T$  is bounded by a constant proportional to the perturbation”. This result, now known as Hyers stability, established a foundational framework for measuring how close approximate solutions are to true solutions.

The theory was significantly generalized by Themistocles M. Rassias in 1978 [19], who introduced a variable error function depending on the norms of variables, leading to the Rassias (HUR) stability. This generalization broadened the scope of stability analysis and inspired extensive research across mathematics and applied sciences [6, 9, 18]. These ideas were later extended to various types of functional equations in [1, 2, 8, 10, 12, 17, 20].

In the context of DE's, Hyers-Ulam stability examines, if a function approximately satisfies a given DE's, Hyers-Ulam stability ensures the existence of an exact solution nearby, with an error bound that depends on the perturbation. This type of stability has been widely studied in ODE and PDE's, as well as in fractional DE's. It finds applications in mathematical physics, Stochastic Processes, control theory, and numerical analysis, where ensuring the robustness of solutions against small errors is crucial. The study of Hyers-Ulam stability has also been extended to various settings, including different types of function spaces and dynamic equations on time scales, making it a versatile tool in modern mathematical analysis. The application of Hyers-Ulam stability to DE's emerged in the latter half of the 20th century. Researchers began investigating whether approximate solutions to ODEs and PDEs guarantee the existence of exact solutions nearby.

A natural extension of Ulam's problem has been developed by considering DE's in place of functional equations. Among the early contributions in this direction are the works of Obłoza [16], who investigated the stability of first-order linear differential equations. His results showed that when a function approximately satisfies a given differential equation, there exists an exact solution that remains within a bounded distance from the approximate one. In general, a differential equation is said to possess Hyers-Ulam stability if, for every positive tolerance, any function that approximately satisfies the equation is close to an exact solution, with the deviation controlled by a bound depending only on the given tolerance. Moreover, this bound approaches zero as the tolerance tends to zero. If this concept is further extended by replacing the constant bound with suitable control functions, independent of the particular solutions involved, then the equation is said to satisfy generalized Hyers-Ulam stability, also known as Hyers-Ulam-Rassias stability.

Following this, Claudi Alsina and Roman Ger [4] investigated the Hyers-Ulam stability of linear differential equations  $y'(t) = y(t)$  in their work. Later, in 1990s-2000s several mathematicians extended these ideas to nonlinear, fractional, and stochastic differential equations, establishing generalized forms of HU stability for a variety of dynamic models. These earlier results were subsequently extended to establish the Hyers-Ulam stability of linear DE's of first, second, and higher in [13, 14, 15]. Recently, considerable attention has been devoted to the study of Ulam stability for linear DE's through different approaches, like transform techniques, special inequalities, Fixed point theorems as well as other analytical methods (one can refer [3, 5, 7, 9, 17]).

## 2 Louis Bachelier's SDE (An Additive Brownian Motion)

Louis Bachelier's (1900) is considered the father of modern stochastic processes in finance. He introduced Brownian motion as a model for stock prices—well before Einstein and Wiener formalized it. He proposed that the evolution of a stock price process  $S_t$  is modeled as

$$dS_t = \vartheta dt + \zeta dW_t$$

where,

- $S_t$  = asset price at time  $t$
- $\vartheta$  = drift (expected return per unit time),
- $\zeta$  = volatility parameter,
- $W_t$  = standard Brownian motion (Wiener process).

This was the first stochastic differential equation (SDE) used to describe random market fluctuations. It laid the mathematical groundwork for later models like Geometric Brownian Motion (used in the Black–Scholes model). It introduces the concept of stochastic drift and diffusion, which are central to modern modelling of real-world dynamic systems.

Naturally, the Bachelier's stochastic differential equation is directly applied to

- Early financial modelling for stock prices.
- Foundation for later models like Geometric Brownian Motion (used in Black–Scholes).
- Used in option pricing, risk analysis, and stochastic control.
- In physics, additive Brownian motion models particle diffusion in fluids.
- In signal processing and control systems, it represents additive Gaussian noise—a key assumption in filters and estimation (like the Kalman filter).

Additive Brownian motion is important because it was the first mathematical model to describe random continuous-time motion, forming the foundation of modern stochastic processes, quantitative finance, and diffusion theory.

Now, we establish the sufficient criteria to ensuring the stability of Louis Bachelier's stochastic differential equation in the sense Ulam. That is, we establish the Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam stability of the Louis Bachelier's Linear SDE (An Additive Brownian Motion) is of the form

$$dS_t = \vartheta dt + \zeta dW_t, \tag{1}$$

for all  $S_t \in C(I)$ ,  $t \in I$  and  $I = [0, T] \subseteq \mathbb{R}$ .

The Hyers–Ulam stability of the Bachelier stochastic differential equation provides a useful theoretical framework for assessing the reliability of stochastic models in carbon control applications. When carbon-related variables such as emission levels or carbon prices are modeled using an additive Brownian motion process, the presence of Hyers–Ulam stability on finite time intervals ensures that small perturbations in the model arising from measurement errors, parameter uncertainty, or external fluctuations lead only to proportionally small deviations in the resulting trajectories. This property supports the robustness of short-term forecasting and policy analysis, where approximate models are often unavoidable. However, due to the non-stationary nature of Brownian motion, the associated stability is not preserved over long time horizons, as the variance of the process increases indefinitely and amplifies accumulated errors. Consequently, while the Bachelier model is suitable for short-term carbon control and pricing strategies, it is less effective for long-term environmental planning. This limitation highlights the importance of incorporating

stabilizing mechanisms, such as mean-reverting dynamics or feedback control, in stochastic models used for sustainable carbon management.

### 3 Basic Definitions

**Definition 3.1 (HU stability of Louis Bachelier's SDE:)** The Additive Brownian Motion (1) is said to have Hyers-Ulam stability, if there exists a constant  $L > 0$  such that the following property holds: For every  $\varepsilon > 0$ , if for every approximate solution  $S_t \in C(I)$  satisfying the inequality  $|dS_t - \vartheta dt - \varsigma dW_t| \leq \varepsilon$ , then there exists an exact solution  $U_t \in C(I)$  satisfying  $dU_t - \vartheta dt - \varsigma dW_t = 0$  such that  $\mathbb{E}[|S_t - U_t|] \leq L \varepsilon$ . The constant  $L$  is called the Hyers-Ulam stability constant for the Additive Brownian motion (1).

**Definition 3.2 (HUR stability of Louis Bachelier's SDE:)** The Additive Brownian Motion (1) is said to have Hyers-Ulam-Rassias stability, if there exists a constant  $L > 0$  such that the following property holds: For every  $\varepsilon > 0$ , if for every approximate solution  $S_t \in C(I)$  and there exists  $\theta_\phi(t) \in C(\mathbb{R}_+, \mathbb{R}_+)$  satisfying the inequality

$$|dS_t - \vartheta dt - \varsigma dW_t| \leq \theta_\phi(t) \varepsilon,$$

then there exists an exact solution  $U_t \in C(I)$  satisfying  $dU_t = \vartheta dt + \varsigma dW_t$  such that

$$\mathbb{E}[|S_t - U_t|] \leq L \theta_\phi(t) \varepsilon.$$

The constant  $L$  is called the HUR stability constant for the Additive Brownian motion (1).

**Definition 3.3 [22]** The Mittag-Leffler function of one parameter is denoted by  $E_\nu(x)$  and defined as

$$E_\nu(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\nu k + 1)}$$

where  $x, \nu \in \mathbb{C}$  and  $\text{Re}(\nu) > 0$ . If we put  $\nu = 1$ , then the above equation becomes

$$E_1(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k + 1)} = \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x.$$

**Definition 3.4 (Mittag-Leffler-HU stability of Louis Bachelier's SDE:)** The Additive Brownian Motion (1) is said to have Mittag-Leffler-HU stability, if there exists a constant  $L > 0$  such that the following property holds: For every  $\varepsilon > 0$ , if for every approximate solution  $S_t \in C(I)$  satisfying the inequality

$$|dS_t - \vartheta dt - \varsigma dW_t| \leq \varepsilon E_\nu(t),$$

where  $E_\nu(t)$  is the Mittag-Leffler function, then there exists  $U_t \in C(I)$  satisfying

$$dU_t - \vartheta dt - \varsigma dW_t = 0$$

such that  $\mathbb{E}[|S_t - U_t|] \leq L \varepsilon E_\nu(t)$ . The constant  $L$  is called the Mittag-Leffler-Hyers-Ulam stability constant for the Additive Brownian motion (1).

**Definition 3.5 (Mittag-Leffler-HUR stability of Louis Bachelier's SDE:)** The Additive Brownian Motion (1) is said to have Mittag-Leffler-HUR stability, if there exists a constant  $L > 0$  such that the following property holds: For every  $\varepsilon > 0$ , if for every approximate solution  $S_t \in C(I)$  and there exists  $\theta_\phi(t) \in C(\mathbb{R}_+, \mathbb{R}_+)$  satisfying the inequality

$$|dS_t - \vartheta dt - \varsigma dW_t| \leq \theta_\phi(t) \varepsilon E_\nu(t),$$

then there exists an exact solution  $U_t \in C(I)$  satisfying  $dU_t = \vartheta dt + \varsigma dW_t$  such that

$$\mathbb{E}[|S_t - U_t|] \leq L \theta_\phi(t) \varepsilon E_\nu(t).$$

The constant  $L$  is called the Mittag-Leffler-HUR stability constant for the Additive Brownian motion (1).

## 4 Main Results - Ulam Stability of Louis Bachelier's SDE

**Theorem 4.1** *The Additive Brownian motion (1) has the HU stability.*

*Proof.* For every  $\varepsilon > 0$ , and assume that for every approximate solution  $S_t \in C(I)$  satisfies the differential inequality

$$|dS_t - \vartheta dt - \zeta dW_t| \leq \varepsilon, \quad (2)$$

for all  $t \in I$ . We demonstrate the existence of a positive constant  $L > 0$  that is independent of  $\varepsilon$  and  $S_t \ni \mathbb{E}[|S_t - U_t|] \leq L \varepsilon$  for some  $U_t \in C(I)$ , satisfying  $dU_t = \vartheta dt + \zeta dW_t$  for all  $t \in I$ . Consider a function  $V_t: (0, \infty) \rightarrow \mathbb{R}$  such that

$$V_t = S_0 + S_t - \int_0^t \vartheta ds - \int_0^t \zeta dW_s.$$

Equivalently, the differential form of the above can be written as

$$\begin{aligned} dV_t &= dS_t - \vartheta dt - \zeta dW_t, \\ \Rightarrow dS_t &= dV_t + \vartheta dt + \zeta dW_t \end{aligned} \quad (3)$$

for all  $t > 0$ . In view of (2), we have  $|dV_t| \leq \varepsilon$ . Define an approximate process  $U_t \in C(I)$  which satisfies the differential model

$$dU_t = [\vartheta + \Delta_1(t)] dt + [\zeta + \Delta_2(t)] dW_t \quad (4)$$

where  $\Delta_1(t)$  and  $\Delta_2(t)$  are small bounded perturbations such that

$$|\Delta_1(t)| \leq \varepsilon_1 \quad \text{and} \quad |\Delta_2(t)| \leq \varepsilon_2 \quad (5)$$

From (3) and (4), we have

$$\begin{aligned} dS_t - dU_t &= \{dV_t + \vartheta dt + \zeta dW_t\} - \{[\vartheta + \Delta_1(t)] dt + [\zeta + \Delta_2(t)] dW_t\} \\ \Rightarrow dS_t - dU_t &= dV_t - \Delta_1(t) dt - \Delta_2(t) dW_t. \end{aligned}$$

Thus, we have

$$d(S_t - U_t) = dV_t - \Delta_1(t) dt - \Delta_2(t) dW_t. \quad (6)$$

Integrating (6), we have

$$S_t - U_t = \int_0^t dV_s - \int_0^t \Delta_1(s) ds - \int_0^t \Delta_2(s) dW_s \quad (7)$$

Taking Expectation and modulus on both sides of (7), we have

$$\mathbb{E}[|S_t - U_t|^2] \leq \mathbb{E} \left[ \left| \int_0^t dV_s \right|^2 \right] + \mathbb{E} \left[ \left| \int_0^t \Delta_1(s) ds \right|^2 \right] + \mathbb{E} \left[ \left| \int_0^t \Delta_2(s) dW_s \right|^2 \right]. \quad (8)$$

Since using (5), we have

$$\mathbb{E} \left[ \left| \int_0^t \Delta_2(s) dW_s \right|^2 \right] \leq \left( \mathbb{E} \left[ \left| \int_0^t \Delta_2(s) dW_s \right|^2 \right] \right)^{\frac{1}{2}} \leq \left( \mathbb{E} \left[ \int_0^t \Delta_2^2(s) \right] \right)^{\frac{1}{2}} \leq \varepsilon_2 \sqrt{T}. \quad (9)$$

Also using (5) and (9) in (8), we have

$$\begin{aligned} \mathbb{E}[|S_t - U_t|^2] &\leq \varepsilon^2 + \varepsilon_1^2 T^2 + \varepsilon_2^2 T \\ \Rightarrow \mathbb{E}[|S_t - U_t|] &\leq \sqrt{\varepsilon^2 + \varepsilon_1^2 T^2 + \varepsilon_2^2 T} \leq L \varepsilon, \end{aligned}$$

where  $L = \sqrt{1 + T + T^2}$  and  $\varepsilon = \max(\varepsilon_1, \varepsilon_2)$ . Hence, by the virtue of the Definition 3.1, the Louis Bachelier's SDE (1) has HU stability.

**Corollary 4.2** *Assume that for every  $S_t \in C(I)$  is an approximate solution of the Additive Brownian Motion (1). For every  $\varepsilon > 0$ , if there exists a real constant  $L > 0$  and  $\theta_\phi(t) \in C(\mathbb{R}_+, \mathbb{R}_+)$  such that the approximate solution  $S_t \in C(I)$  satisfying the inequality*

$$|dS_t - \vartheta dt - \zeta dW_t| \leq \theta_\phi(t) \varepsilon,$$

then there exists an exact solution  $U_t \in C(I)$  satisfying the motion  $dU_t = \vartheta dt + \zeta dW_t$  such that  $\mathbb{E}[|S_t - U_t|] \leq L \theta_\phi(t)\varepsilon$ .

**Theorem 4.3** *The Louis Bachelier's SDE (1) is the Mittag-Leffler-HU stable.*

*Proof.* For every  $\varepsilon > 0$ , and assume that  $S_t \in C(I)$  be an approximate solution of (1), which satisfies the inequality

$$|dS_t - \vartheta dt - \zeta dW_t| \leq \varepsilon E_\alpha(t), \tag{10}$$

for all  $t \in I$ , where  $E_\alpha(t)$  is the Mittag-Leffler function. We wish to prove that there exists real constant  $L > 0$  which is independent of  $\varepsilon$  and  $S_t$  such that

$$\mathbb{E}[|S_t - U_t|] \leq L \varepsilon E_\alpha(t)$$

for some  $U_t \in C(I)$  satisfies  $dU_t = \vartheta dt + \zeta dW_t$  for all  $t \in I$ . Let  $V_t: (0, \infty) \rightarrow \mathbb{R}$  such that

$$V_t = S_0 + S_t - \int_0^t \vartheta ds - \int_0^t \zeta dW_s.$$

Equivalently, the differential form of the above can be written as

$$dS_t = dV_t + \vartheta dt + \zeta dW_t \tag{11}$$

for all  $t > 0$ . By (10), we have  $|dV_t| \leq \varepsilon E_\alpha(t)$ . Define an approximate process  $U_t \in C(I)$  which satisfies the Bachelier's SDE model

$$dU_t = [\vartheta + \Delta_1(t)] dt + [\zeta + \Delta_2(t)] dW_t \tag{12}$$

where  $\Delta_1(t)$  and  $\Delta_2(t)$  are small bounded perturbations such that

$$|\Delta_1(t)| \leq \varepsilon_1 E_\alpha(t) \quad \text{and} \quad |\Delta_2(t)| \leq \varepsilon_2 E_\alpha(t). \tag{13}$$

Using (11) and (12) we have

$$d(S_t - U_t) = dV_t - \Delta_1(t) dt - \Delta_2(t) dW_t. \tag{14}$$

Integrating (14) and taking Expectation and modulus on each sides of the resulting equation,

$$\mathbb{E}[|S_t - U_t|^2] \leq \mathbb{E} \left[ \left| \int_0^t dV_s \right|^2 \right] + \mathbb{E} \left[ \left| \int_0^t \Delta_1(s) ds \right|^2 \right] + \mathbb{E} \left[ \left| \int_0^t \Delta_2(s) dW_s \right|^2 \right] \tag{15}$$

By using (13), we have

$$\mathbb{E} \left[ \left| \int_0^t \Delta_2(s) dW_s \right|^2 \right] \leq \left( \mathbb{E} \left[ \left| \int_0^t \Delta_2(s) dW_s \right|^2 \right] \right)^{\frac{1}{2}} \leq \left( \mathbb{E} \left[ \int_0^t \Delta_2^2(s) \right] \right)^{\frac{1}{2}} \leq \varepsilon_2 E_\alpha(t) \sqrt{T} \tag{16}$$

Then, with the help of (13) and (16), the inequality (15) becomes

$$\begin{aligned} \mathbb{E}[|S_t - U_t|^2] &\leq \varepsilon^2 E_\alpha^2(t) + \varepsilon_1^2 E_\alpha^2(t) T^2 + \varepsilon_2^2 E_\alpha^2(t) T \\ \Rightarrow \mathbb{E}[|S_t - U_t|] &\leq \varepsilon E_\alpha(t) \sqrt{1 + T + T^2} \leq L \varepsilon E_\alpha(t), \end{aligned}$$

where  $L = \sqrt{1 + T + T^2}$  and  $\varepsilon = \max(\varepsilon_1, \varepsilon_2)$ . Hence, by the virtue of the Definition 3.4, the Louis Bachelier's SDE (1) has Mittag-Leffler-HU stability.

**Corollary 4.4** *For every  $\varepsilon > 0$ , if there exists a real constant  $L > 0$  and  $\theta_\phi(t) \in C(\mathbb{R}_+, \mathbb{R}_+)$  such that the approximate solution  $S_t \in C(I)$  satisfying the inequality*

$$|dS_t - \vartheta dt - \zeta dW_t| \leq \theta_\phi(t) \varepsilon E_\alpha(t),$$

then  $\exists$  an exact solution  $U_t \in C(I)$  satisfying the motion  $dU_t = \vartheta dt + \zeta dW_t$  such that  $\mathbb{E}[|S_t - U_t|] \leq L \theta_\phi(t)\varepsilon E_\alpha(t)$ .

## 5 Discussion: Application of the SDE

In carbon-related systems (like carbon price, emissions level, or pollution index), uncertainty plays a major role. So we often model such quantities using SDEs like:

$$dC_t = \vartheta dt + \zeta dW_t \quad (17)$$

where,

- $C_t$  = carbon price / emissions level
- $\vartheta$  = average trend (policy effect, growth/decline),
- $\zeta$  = uncertainty (market shocks, climate variability),
- $W_t$  = Standard Brownian motion (Wiener process).

This is exactly the Bachelier's-type model (additive noise). Using our results, we can easily prove the following results.

**Result 4.1** *The Additive Brownian motion (17) has the HU stability.*

**Result 4.2** *The Additive Brownian motion (17) has the HUR stability.*

**Result 4.3** *The Additive Brownian motion (17) has the Mittag-Leffler-HU stability.*

**Result 4.4** *The Additive Brownian motion (17) has the Mittag-Leffler-HUR stability.*

If our carbon model is slightly wrong (measurement errors, policy uncertainty), will the predictions still be reliable?

If the system is HU stable (on finite time), Small modelling errors leads small prediction errors and the Policies based on the model are robust in the short term. If not globally stable: Errors grow over time, Long-term predictions become unreliable.

Practically, in carbon control, Short-term policy is stable in Carbon pricing (tax or cap-and-trade) decisions, Emission targets over a few years, Forecasting near-term pollution levels. Hyers–Ulam stability ensures that Small uncertainty in data and small deviation in outcomes.

For long-term policy it is not stable in Climate projections over decades, Long-term carbon market predictions. Because of Brownian noise, uncertainty accumulates and Predictions drift away. In control theory, we want systems that are: Stable, Predictable and Robust. But Bachelier's model is not inherently stabilizing (no feedback, no mean reversion). So in carbon control, we often improve it using the mean reverting models

$$dC_t = \theta (\alpha - C_t) dt + \zeta dW_t$$

These are more suitable because they pull the system back to a target level and they improve long-term stability. The Hyers–Ulam stability of Bachelier's SDE tells us

- The model is reliable for short-term carbon control decisions
- But not sufficient for long-term climate control
- Therefore, policy models must include stabilizing mechanisms (feedback, regulation, and mean reversion).

Hyers–Ulam stability confirms robustness of carbon models, Bachelier's model gives the short-term uncertainty modelling and Carbon control needs stronger (stabilizing) models for long-term planning.

## 6 Conclusion

The study of the Bachelier's stochastic differential equation indicates that the model satisfies Hyers–Ulam stability over finite time intervals. This means that if a process approximately fulfils the given SDE with a small error, then there exists an exact solution that stays close to it, and the difference can be bounded in terms of the size of the perturbation. However, this stability does not extend uniformly over an infinite time horizon. The random fluctuations introduced by Brownian motion cause the variance of the solution to increase with time, and consequently, even small errors may accumulate and grow. Because of this, the stability bounds depend on the length of the time interval and become larger as time increases. In brief, the Bachelier's model demonstrates local (finite-time) Hyers-Ulam stability, but it does not exhibit global stability due to the unbounded growth inherent in its stochastic component.

The connection between Hyers–Ulam stability and carbon control can be effectively understood through the analysis of the Bachelier's SDE which serves as a simple model for uncertain quantities such as carbon prices or emission levels. In this framework, the presence of Hyers–Ulam stability on finite time intervals implies that small perturbations in the model - arising from data inaccuracies, environmental variability, or policy uncertainties - result only in proportionally small deviations in the system's behaviour. This provides a theoretical justification for the reliability and robustness of short-term carbon control strategies, where approximate models are commonly employed. However, the additive nature of the noise in the Bachelier's model leads to an unbounded growth of variance over time, meaning that the stability is not preserved in the long run. As time progresses, even minor errors accumulate and may significantly affect the system's trajectory, reducing the reliability of long-term predictions. This limitation indicates that, while the model is appropriate for short-term forecasting and policy implementation, it is not sufficient for long-term carbon management and climate planning.

Therefore, from a control perspective, it becomes necessary to adopt more advanced stochastic models that incorporate stabilizing features such as feedback mechanisms or mean-reverting behaviour. Such models can better regulate the system around desired targets and ensure improved long-term stability. In summary, the Hyers–Ulam stability of the Bachelier's model highlights its usefulness for short-term carbon control while also emphasizing the need for enhanced modelling approaches in long-term environmental sustainability.

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