Universal critical dynamics near the chiral phase transition and the QCD critical point

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Abstract. We use a novel real-time formulation of the functional renormalization group (FRG) for dynamical systems with reversible mode couplings to study Model G and H, which are the conjectured dynamic universality classes of the two-flavor chiral phase transition and the QCD critical point, respectively. We compute the dynamic critical exponent in both models in spatial dimensions $2 < d < 4$. We discuss qualitative commonalities and differences in the non-perturbative RG flow, such as weak scaling relations which hold in either case versus the characteristic strong scaling of Model G which is absent in Model H. For Model G, we also extract a novel dynamic scaling function that describes the universal momentum and temperature dependence of the diffusion coefficient of iso-(axial-)vector charges in the symmetric phase.

1 Introduction

Understanding the real-time dynamics of hot and dense QCD matter in relativistic heavy-ion collisions from first principles is a challenging task. Near a second-order phase transition the divergent correlation time due to critical slowing down entails that the dynamics of the long-wavelength modes becomes universal. Corresponding *dynamic* universality classes were classified by Halperin and Hohenberg [1].

The chiral phase transition in the two-flavor chiral limit is expected to be second order and to fall into the $O(4)$ universality class as long as the axial $U(1)_A$ symmetry is not effectively restored at the critical temperature [2]. Since the physical up and down quark masses are much smaller than other typical QCD scales, it is not unreasonable to expect dynamical effects from the $O(4)$ chiral phase transition as the fireball passes the chiral crossover region [3]. An argument by Rajagopal and Wilczek suggests that the corresponding dynamic universality class is the one of Model G, i.e., a Heisenberg antiferromagnet [4].

A true second-order phase transition in QCD with physical quark masses would be present in the form of the conjectured Z_2 critical point at finite baryon chemical potential. In this case, an argument by Son and Stephanov suggests that the corresponding dynamic universality class is the one of Model H, i.e., the liquid-gas critical point in a pure fluid [5].

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For static critical phenomena, the functional renormalization group (FRG) can produce quantitative results for, e.g., critical exponents and amplitude ratios. On the other hand, dynamic critical phenomena require a real-time formulation of the FRG. While the real-time FRG has been successfully employed in the description of the purely relaxational Models A, B, & C, Models G and H, on the other hand, are more subtle since they involve 'reversible mode couplings'. Here, we summarize some results of our previous work [6, 7] and discuss how a real-time FRG flow which preserves the associated symmetries can be formulated.

2 Real-time FRG approach to Model G/H

We consider the Landau-Ginzburg-Wilson (LGW) free energy

$$F = \int d^d x \left\{ \frac{1}{2}(\partial^i \phi_a)(\partial^i \phi_a) + \frac{m^2}{2}\phi_a\phi_a + \frac{\lambda}{4!N}(\phi_a\phi_a)^2 \right\} \tag{1}$$

with $m^2 < 0$ to spontaneously break the $O(N)$ symmetry of the order parameter, and a positive quartic coupling $\lambda > 0$ for stability at large field values.

In Model G, the order parameter is not conserved, but couples reversibly to an anti-symmetric tensor of charge densities n_{ab}, which are the generators of $O(N)$ transformations. Since their static fluctuations are non-critical, they enter the LGW free energy (1) as $F \to F + \int d^d x\, n_{ab} n_{ab}/4\chi$. The equations of motion of Model G can be written as [4]

$$\frac{\partial \phi_a}{\partial t} = -\Gamma^\phi \frac{\delta F}{\delta \phi_a} + \frac{g}{2}\{\phi_a, n_{bc}\}\frac{\delta F}{\delta n_{bc}} + \theta_a\,, \tag{2a}$$

$$\frac{\partial n_{ab}}{\partial t} = \gamma \vec{\nabla}^2 \frac{\delta F}{\delta n_{ab}} + g\{n_{ab}, \phi_c\}\frac{\delta F}{\delta \phi_c} + \frac{g}{2}\{n_{ab}, n_{cd}\}\frac{\delta F}{\delta n_{cd}} + \zeta_{ab}\,, \tag{2b}$$

where θ_a and ζ_{ab} are Langevin noise terms whose variances set by the fluctuation-dissipation relation, and $\{\cdot,\cdot\}$ denotes the Poisson bracket, which is explicitly spelled out in Ref. [7].

In Model H, the order parameter has $N = 1$ component, is conserved, and couples reversibly to the transverse part $\vec{j}$ of the momentum density. The equilibrium fluctuations of $\vec{j}$ are non-critical, which means that $\vec{j}$ appears in (1) only quadratically, $F \to F + \int d^d x\, \vec{j}^2/2\rho$, where ρ is the mass density. The equations of motion of Model H are given by

$$\frac{\partial \phi}{\partial t} = \sigma \vec{\nabla}^2 \frac{\delta F}{\delta \phi} + g\{\phi, \vec{j}\}\cdot\frac{\delta F}{\delta \vec{j}} + \theta\,, \tag{3a}$$

$$\frac{\partial j_l}{\partial t} = \mathcal{T}_{lm}\left[\eta \vec{\nabla}^2 \frac{\delta F}{\delta j_m} + g\{j_m, \phi\}\frac{\delta F}{\delta \phi} + g\{j_m, j_n\}\frac{\delta F}{\delta j_n}\right] + \xi_l\,, \tag{3b}$$

where $\mathcal{T}$ is a transverse projector, and θ and ξ_l are again Langevin noise terms.

The basis for the real-time functional renormalization group is the Martin-Siggia-Rose (MSR) path-integral reformulation of the stochastic equations (2) and (3). Importantly, the structure of the Poisson-bracket terms imply an extended temporal gauge symmetry of the MSR action [6], which is not present in the relaxational Models A, B & C. This extended symmetry can be preserved during the FRG flow by introducing the bilinear regulator term at the level of the LGW free energy (1), rather than in the MSR action. While this makes no difference in the relaxational Models A, B & C, the Poisson brackets in (2) and (3) entail that – at the level of the MSR action – the regulator couples to *composite* response fields. Introducing the regulator in this way ensures that the flow of free energy is independent of dynamics, and that the mode-coupling constant g is protected from loop corrections.

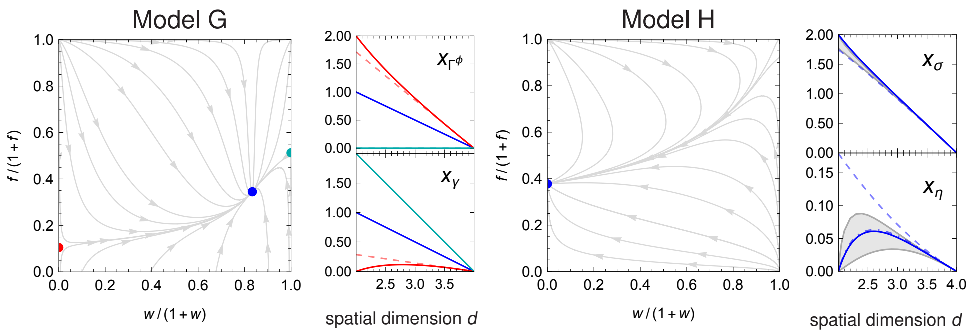

Figure 1. FRG flow diagrams of 3d Model G and H. Model G admits one attractive strong-scaling fixed point (blue) at $(w^*, f^*) \approx (4.998, 0.527)$, and two unstable weak-scaling fixed points (red and green) at $w^* = 0, \infty$. Model H admits one attractive weak-scaling fixed point (blue) at $w^* = 0$. Critical exponents of the kinetic coefficients are plotted as a function of spatial dimension $2 < d < 4$ in matching colors. Here, solid lines indicate our FRG results with ϕ^4-truncation, and dashed lines results from the perturbative ϵ-expansion. Gray bands show FRG results with the extended local-potential approximation (LPA') of the free energy, with the upper and lower bounds corresponding to the field expansion points $\phi = 0$ and the scale-dependent minimum of the effective potential. Figure taken from Ref. [7].

3 Results

We truncate the FRG flow by promoting the couplings m^2 and λ in LGW free energy (1) to depend on the FRG scale (which we call 'ϕ^4-truncation'), as well as the kinetic coefficients Γ^ϕ and γ in (2) and σ and η in (3). Second-order phase transitions correspond to fixed points of the FRG flow. Suitable dimensionless combinations of the kinetic coefficients are given by

$$w_G \equiv \chi \frac{\Gamma^\phi_k}{\gamma_k}, \quad f_G \equiv \frac{d\,\Omega_d\, g^2 T}{(2\pi)^d} \frac{k^{d-4}}{\Gamma^\phi_k \gamma_k}, \qquad w_H \equiv \rho \frac{\sigma_k k^2}{\eta_k}, \quad f_H \equiv \frac{d\,\Omega_d\, g^2 T}{(2\pi)^d} \frac{k^{d-4}}{\sigma_k \eta_k}. \tag{4}$$

Their FRG flow is shown in Fig. 1. At the fixed points, the kinetic coefficients behave as power laws with the FRG scale, $\Gamma^\phi_k \sim k^{-x_{\Gamma^\phi}}$, $\gamma_k \sim k^{-x_\gamma}$, $\sigma_k \sim k^{-x_\sigma}$ and $\eta_k \sim k^{-x_\eta}$.

Qualitatively, the finite fixed-point value of $0 < f^*_{G/H} < \infty$ implies the scaling relation $x_{\Gamma^\phi} + x_\gamma = 4 - d - \eta_\perp$ in Model G, and $x_\sigma + x_\eta = 4 - d - \eta_\perp$ in Model H (here including the anomalous dimension $\eta_\perp$ of the order parameter). On the other hand, the fixed-point value w^*_G is finite only at the strong-scaling fixed point of Model G, where it implies strong-scaling behavior with $z_\phi = z_n = d/2$. This relation is absent at the other fixed points, which either have $w^*_{G/H} = 0$ or $w^*_{G/H} = \infty$. In the phenomenologically relevant case of $d = 3$ spatial dimensions we obtain the values $x_\sigma \approx 0.949$ and $x_\eta \approx 0.051$ in Model H, which correspond to the dynamic critical exponent $z_\phi = d + x_\eta \approx 3.051$. We observe for $d = 2$ that the critical exponent of the shear viscosity vanishes, $x_\eta = 0$.

In Ref. [6], we have supplemented our truncation of Model G by momentum dependence of iso-(axial-)vector charge mobility $\gamma_k \to \gamma_k(\vec{p})$. The dynamic scaling hypothesis for the associated diffusion coefficient $D_n(\vec{p}) \equiv \gamma(\vec{p})/\chi$ implies the scaling form (at $k = 0$)

$$D_n(\vec{p}, \tau) = D_n^+ \tau^{-\nu x_\gamma} \mathcal{L}(p\xi(\tau)) \tag{5}$$

for its momentum and temperature dependence in the scaling regime. Here, $\tau \equiv (T - T_c)/T_c$ is the reduced temperature, and $\xi(\tau) = 1/m_{k=0}$ is the correlation length. As discussed above, we have $x_\gamma = 2 - z_n = 2 - d/2$ at the strong-scaling fixed point. By tuning the temperature

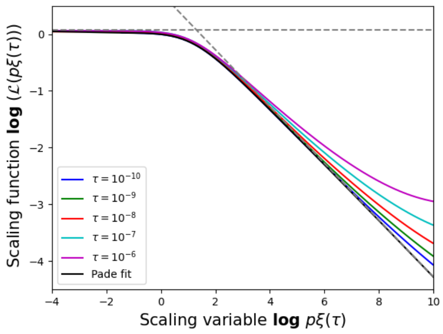

Figure 2. By tuning the temperature close to the critical temperature from above, and rescaling the resulting diffusion coefficient $D_n(\vec{p})$ at $k = 0$ according to (5), we find a scaling function $\mathcal{L}(x)$ which only depends on the scaling variable $x \equiv p\xi(\tau)$. Figure taken from Ref. [6].

close to the critical temperature from above, and rescaling the resulting $D_n(\vec{p}, \tau)$ according to (5), the curves collapse onto a scaling function $\mathcal{L}(x)$, which is shown in Fig. 2.

4 Outlook

The present formulation of the real-time FRG lays the foundation for more sophisticated truncation schemes in the future. For example, by including momentum dependence in the kinetic coefficients of Model H one can compute scaling functions which go beyond the commonly employed Kawasaki approximation. On the other hand, by including external fields in Model G (which correspond to finite current quark masses in QCD), one can study the influence of critical dynamics on hot QCD matter at the physical point within a low-energy effective theory. Besides dynamic critical phenomena, another application of the present formalism is fluctuating hydrodynamics, where the real-time FRG can be used to study the influence of fluctuations on transport coefficients non-perturbatively.

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