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Simulating stochastic fluid dynamics

Chandrodoy Chattopadhyay[1], *Josh* Ott[2], *Thomas* Schäfer[1,*], and *Vladimir* Skokov[1]

[1]Department of Physics, North Carolina State University, Raleigh, NC 27695, USA

[2]Center for Theoretical Physics – a Leinweber Institute, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

Abstract. We present simulations of the real time dynamics of a fluid in the vicinity of a critical endpoint in the phase diagram. The relevant hydrodynamic theory is known as model H, and it is expected to describe the long-distance dynamics of QCD matter in the vicinity of a possible critical endpoint in the QCD phase diagram.

1 Introduction

One of the goals of the heavy ion program at RHIC, the LHC, and future facilities at lower energy is to find direct evidence for a phase transition between a quark-gluon phase and a hadronic phase of QCD. Two ideas that have been discussed is to observe fluctuations signatures associated with the chiral crossover transition at zero chemical potential, or to observe critical fluctuations related to a possible endpoint of a first order transition at non-zero chemical potential.

The critical dynamics in the vicinity of a phase transition is described by fluid dynamic theories that contain stochastic terms in addition to the well-known gradient expansion. These stochastic terms ensure that fluctuation-dissipation relations are satisfied, in particular in a regime where fluctuations are not small. A theory that describes the universal critical dynamics near an endpoint in the universality class of the liquid-gas endpoint is model H [1]. This theory is expected to describe the dynamics of a QCD fluid in the vicinity of a possible endpoint of a first order transition in the phase diagram [2].

There is a significant amount of literature on the dynamics of fluctuations in QCD [3], but model H has never been numerically simulated. This is the case because there are a number of difficulties that need to be overcome. Stochastic fluid dynamic models have fluctuations on all scales, ranging from the size of the system to the microscopic scale at which the theory is discretized. These fluctuations lead to ultra-violet divergences that have to be renormalized, and the regularization and renormalization procedure has to be compatible with fluctuation-dissipation relations and stability requirements. In the following we briefly explain a new approach to this problem, and summarize the results obtained in our recent work [4, 5].

*e-mail: tmschaef@ncsu.edu

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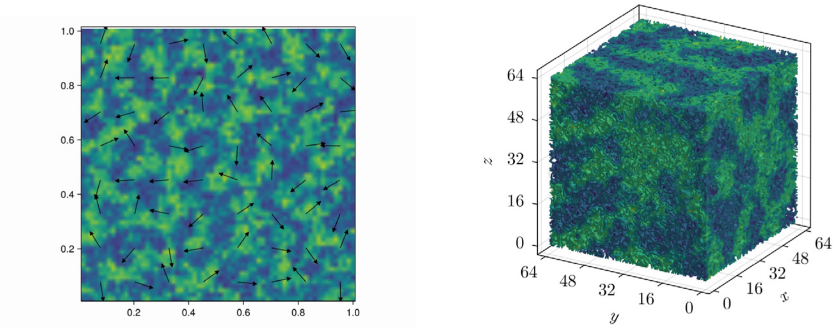

Figure 1. Left panel: Order parameter ϕ (color coded) and fluid momentum $\vec{\pi}$ (arrows) configuration in a stochastic two-dimensional fluid. Right panel: Order parameter configuration in a three dimensional fluid.

2 Model H

Model H describes the interaction of an order parameter density ϕ with the momentum density $\vec{\pi}$ of the fluid. This equations of motion are given by [1]

$$\partial_t\phi = \Gamma\,\nabla^2\left(\frac{\delta\mathcal{H}}{\delta\phi}\right) - (\nabla_i\phi)\,\frac{\delta\mathcal{H}}{\delta\pi_i^T} + \zeta, \tag{1}$$

$$\partial_t\pi_i^T = \eta\nabla^2\left(\frac{\delta\mathcal{H}}{\delta\pi_i^T}\right) + P_{ij}^T\left[(\nabla_j\phi)\,\frac{\delta\mathcal{H}}{\delta\phi}\right] - P_{ij}^T\left[\nabla_k\left(\pi_j^T\frac{\delta\mathcal{H}}{\delta\pi_k^T}\right)\right] + \xi_i\,. \tag{2}$$

For the purpose of describing the dynamics near a liquid-gas endpoint we can take ϕ to be the specific entropy s/n of the fluid [6]. Γ is the thermal conductivity and η is the shear viscosity. The transverse projection operator is given by $P_{ij}^T = \delta_{ij} - \nabla_i\nabla_j/\nabla^2$ and the transverse momentum density is $\pi_i^T = P_{ij}^T\pi_j$. The effective Hamiltonian is given by

$$\mathcal{H} = \int d^dx\left[\frac{1}{2\rho}(\pi_i^T)^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{4}\lambda\phi^4 - h\phi\right], \tag{3}$$

where ρ is the mass density, m is the inverse correlation length, λ is a non-linear self-coupling, and h is an external field. At the mean field level this Hamiltonian has a critical point at $h = 0$ and $m^2 = 0$. When fluctuations are included the critical value of m^2 at $h = 0$ has to be determined numerically. In order to describe a critical endpoint in the QCD phase diagram the parameters m^2, h, and λ will have to be mapped onto the chemical potential-temperature (μ, T) plane in QCD, along the lines described in [7]. The noise fields ζ and ξ_i are random variables constrained by fluctuation-dissipation relations. The correlation functions of the noise fields are given by

$$\langle\zeta(t,\vec{x})\zeta(t',\vec{x}')\rangle = -2T\,\Gamma\,\nabla^2\delta(\vec{x}-\vec{x}')\delta(t-t')\,, \tag{4}$$

$$\langle\xi_i(t,\vec{x})\xi_j(t',\vec{x}')\rangle = -2T\,\eta\,P_{ij}^T\nabla^2\delta(\vec{x}-\vec{x}')\delta(t-t')\,. \tag{5}$$

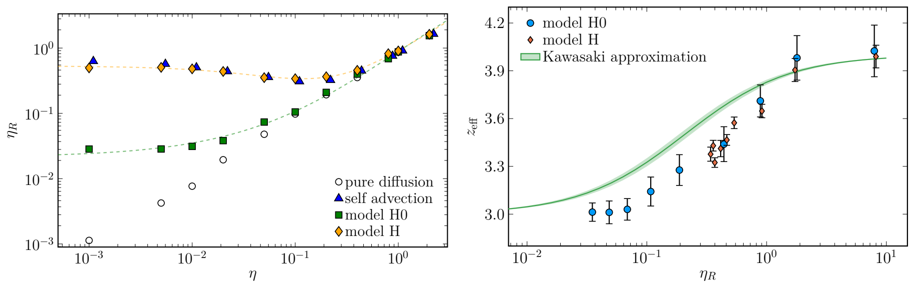

Figure 2. Left panel: Physical shear viscosity as a function of the bare viscosity in a non-critical fluid. Several truncations are shown: Model H0 (no self-advection), self-advection only, and pure diffusion (no interactions). Right panel: Dynamical critical exponent z extracted from a scaling analysis of the order parameter correlation function in a finite volume as a function of the physical viscosity.

3 Numerical Methods

The main idea underlying the numerical algorithm is to combine the dissipative and stochastic updates into a single Metropolis update. The update for the order parameter field is

$$\begin{aligned} \phi^{trial}(\vec{x}, t+\Delta t) &= \phi(\vec{x}, t) + q_\mu\,, \\ \phi^{trial}(\vec{x}+\hat{\mu}, t+\Delta t) &= \phi(\vec{x}+\hat{\mu}, t) - q_\mu\,, \end{aligned} \qquad q_\mu = \sqrt{2\Gamma T(\Delta t)}\,\xi\,, \tag{6}$$

where ξ is a Gaussian random variable with unit variance and $\hat{\mu}$ is an elementary lattice vector in the direction $\mu = 1, \ldots, d$. The update is accepted with probability $min(1, e^{-\Delta\mathcal{H}/T})$. Note that this algorithm is automatically conserving. This algorithm is based on the observation that the average update $\langle[\phi(\vec{x}, t+\Delta t) - \phi(\vec{x}, t)]\rangle$ realizes the diffusion equation, and the second moment $\langle[\phi(\vec{x}, t+\Delta t) - \phi(\vec{x}, t)]^2\rangle$ reproduces the noise term, see [8]. We follow the same procedure for $\vec{\pi}$. We perform a conserving Metropolis update for all components of the momentum density, and the apply a transverse projection operator.

In principle, the advection step can be discretized using standard methods in computational fluid dynamics. However, some attention has to be paid to the fact that the fields exhibit large fluctuations on the scale of the lattice spacing, even as the lattice spacing is taken to zero. This means that we may encounter large violations of conservation laws that follow from the ideal equations of motion in the continuum limit. In practice we have used a discretization method that employs "skew-symmetrized derivatives" which has been developed for the purpose of simulating incompressible turbulence [9].

4 Results and outlook

We have performed simulations of critical and non-critical fluid in box of volume L^3, see Fig. 1 We have verified that m^2 can be tuned to a critical point at which the simulation reproduces the static critical exponents of the 3d Ising Model. The simplest dynamic phenomenon that we have studied is the renormalization of the shear viscosity η of a non-critical fluid, see the left panel of Fig. 2. Perturbative calculations suggest that there is a UV sensitive renormalization of η that is inversely proportional to the bare value of η, an effect sometimes called the "stickiness of sound" [10]. We clearly observe this effect, and we find it to be dominated by the self-advection of the momentum density. This effect is included in the full model H simulation, but dropped in the truncation denoted as model H0.

We have also measured the dynamic critical exponent z. This quantity is a measure of critical slowing-down, the increase of the relaxation time $\tau \sim \xi^z$ as the correlation length ξ increases. We have measured this effect directly at the critical point by varying the system size L, which limits the correlation length in a finite volume. We observe a value $z \simeq 3.01$ which is clearly different from mean-field behavior, but consistent with predictions from the epsilon-expansion. We also observe that in any finite system there is a crossover from model B like behavior $z \simeq 4$ to model H scaling $z \simeq 3$ as the viscosity is reduced. Finally, we observe evidence for universality: The scaling exponent in model H and model H0 is the same when plotted as a function of the physical viscosity (see Fig. 2 right panel).

We have demonstrated that we can also measure more complicated observables, such as the relaxation rate of higher moments of the order parameter. Future work will focus on trying to model realistic systems, by embedding the framework discussed here in an expanding relativistic fluid.

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