EPJ Web of Conferences **364**, 15004 (2026) https://doi.org/10.1051/epjconf/202636415004
Quark Matter 2025

Finite-volume analysis and universal scaling signatures near the chiral phase transition in (2+1)-flavor QCD

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Abstract. For quantifying the universal properties of the chiral phase transition in QCD through numerical calculations on a discrete space-time lattice, one needs to perform controlled extrapolations to the continuum and infinite-volume limits followed by an extrapolation to the limit of massless light quarks. We discuss here, the results on the latter two limits at still finite lattice spacings. We use here for chiral symmetry breaking, an improved order parameter free of additive and multiplicative divergences and we analyse its volume and quark mass dependence. Comparing to the expected universal behavior in the chiral limit, we quantify deviations from the universal finite-size scaling behavior as function of the light to strange quark mass ratio.

1 Introduction

In order to quantify universal (e.g. the critical exponents) and non-universal (e.g. the phase transition temperature) properties of the chiral phase transition in Quantum Chromodynamics (QCD), a carefully controlled extrapolation to the (i) continuum, (ii) infinite-volume and (iii) chiral limits is needed. This has been pursued in the first determination of the chiral phase transition temperature in (2+1)-flavor QCD with smaller than physical light quark masses and a physical value for the strange quark mass [1]. Results from an analysis of pseudo-critical temperatures in (2+1)-flavor QCD obtained with the Highly Improved staggered quark (HISQ) action are shown in Fig. 1. This analysis made use of known finite-volume scaling functions of the 3-d, $O(4)$ universality class [2] to perform joint infinite-volume and continuum limit extrapolations of pseudo-critical temperatures in calculations with light (m_ℓ) to strange (m_s) quark mass ratios $H = m_\ell/m_s \geq 1/160$. In order to allow for a direct determination of universal critical exponents and thus derive the underlying universality class for the chiral phase transition, one needs to extend these

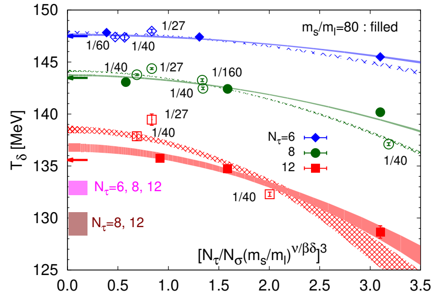

Figure 1. Volume and quark mass extrapolation of pseudo-critical temperatures in (2+1)-flavor QCD with HISQ action (taken from [1]).

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calculations to smaller H and perform a more systematic analysis of finite-volume corrections that allows to identify the unique universal finite-volume contributions to physical observables. We present here first results for finite-volume scaling corrections to the chiral order parameter on lattices with fixed temporal extent, $N_\tau = 8$, varying the spatial extent N_σ. We compare our results with expected universal critical behavior in the 3-d, $O(2)$ universality class and quantify the magnitude of sub-leading corrections.

2 Volume dependence of the order parameter

The starting point of this work encompasses the multiplicatively renormalised versions of the 2-flavor light quark chiral condensate, $M_\ell = m_s \langle \bar\psi \psi \rangle / f_K^4$, and the total chiral susceptibility $\chi_\ell = m_s \partial M_\ell / \partial m_\ell$. Here f_K denotes the kaon decay constant. To attain a well-defined improved order parameter M in the continuum limit, which in the chiral limit also has a straightforward interpretation in terms of universal scaling functions, we use the so-called subtracted order parameter [3, 4], where additive ultraviolet divergent contributions to M_ℓ are eliminated,

$$M(T, H, L) = M_\ell(T, H, L) - H \chi_\ell(T, H, L) \; . \tag{1}$$

Here $L = N_\sigma / N_\tau$ denotes the aspect ratio of a $N_\sigma^3 \cdot N_\tau$ lattice.

At non-vanishing lattice spacing 'a', the temperature and the volume are given by $T = (N_\tau a)^{-1}$ and $V = (N_\sigma a)^3$. We also note that contributions linear in H, present otherwise in the regular and singular parts of M_ℓ, are explicitly cancelled in the definition of the order parameter M given in Eq.(1), consequently reducing the non-universal regular part. In the vicinity of the chiral critical point $(t, h, l) = (0, 0, 0)$, one has,

$$M(T, H, L) = h^{1/\delta} f_{G\chi}(z, z_L) + M_{sub} \; , \tag{2}$$

where $f_{G\chi} = f_G$ - f_χ with f_G, f_χ the respective scaling functions of M_ℓ and χ_ℓ, and M_{sub} refers to sub-dominant contributions arising from corrections-to-scaling as well as regular terms. The scaling variables z, z_L are as follows,

$$z = t\, h^{-1/\beta\delta}, \quad z_L = l\, h^{-\nu_c} \; , \tag{3}$$

where $t = (T/T_c - 1)/t_0$ with chiral phase transition temperature T_c, $h = H/h_0$, $l = l_0/L$, along with the critical exponents β, δ and ν of the associated 3-d universality class with $\nu_c = \nu/\beta\delta = (1 + 1/\delta)/3$. Further notations and detailed discussions are outlined in [5–7]. The dominant universal term of Eq. (2) can be directly exploited to derive the underlying universality class since no conclusive first order evidence has been found yet here in (2+1)-flavor QCD on lattice [8]. However, as mentioned before one requires for this purpose, a detailed analysis at small quark mass and on large lattices to control the chiral and infinite-volume limits. This systematically diminishes contributions from M_{sub}, given in Eq.(2), as well as finite-volume corrections to the underlying universal behavior, thereby augmenting the latter. We illustrate some preliminary results in the next section.

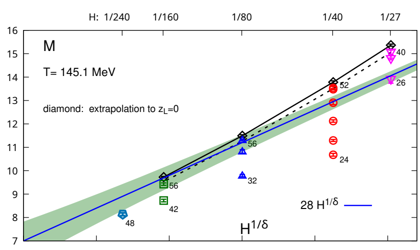

Figure 2. Results of order parameter M as function of $H^{1/\delta}$ at $T = 145.1$ MeV obtained on several $N_\sigma^3 \cdot 8$ lattices, with varying N_σ. Here, $\delta = 4.7798$ and $H = m_\ell/m_s$. Black diamond points show extrapolated infinite-volume limit ($z_L = 0$) values for different H, or light quark masses m_ℓ. The green band shows a 0.5 MeV error band on our new preliminary estimate for T_c on $N_\tau = 8$ lattices and the new, preliminary slope of the blue line $M = m_0 H^{1/\delta}$ is taken to be $m_0 = 28$, which is $\sim 10\%$ smaller than the value for $m_0 = (1 - 1/\delta)\, h_0^{-1/\delta}$, obtained in [5].

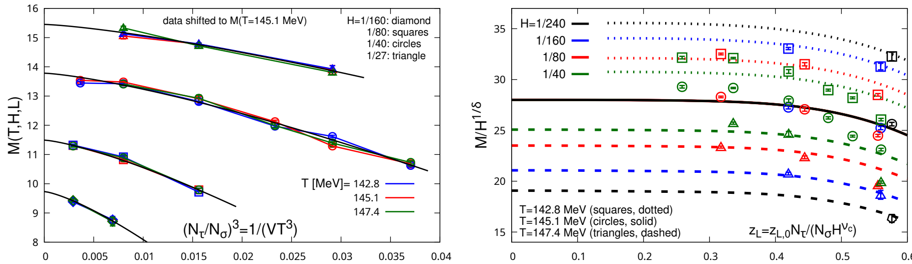

Figure 3. *Left:* Finite-volume dependence of the order parameter M. Shown are results for M versus the inverse volume normalized by T^3 obtained on lattice with temporal extent $N_\tau = 8$ at four values of the quark mass ratio H and three temperature values in the vicinity of $T_c \simeq 145$ MeV. Data obtained at $T = 142.8$ MeV and $T = 147.4$ MeV have been shifted by single, constant values. *Right:* The scaled order parameter $M/H^{1/\delta}$ shown as a function of the finite-volume scaling variable z_L. Lines show the scaling function $f_{G\chi}(z, z_L)$ for three temperatures using preliminary results for the non-universal scale parameters, $T_c = 145.1$ MeV, $z_0 = h_0^{1/\beta\delta}/t_0 = 1.52$ and $z_{L,0} = l_0\, h_0^{\nu_c} = 0.38$.

3 Universal finite-volume effects

In this work, we have used tree-level Symanzik-improved gauge and HISQ actions implemented in *SIMULATeQCD* [9, 10] for numerical calculations. While keeping the strange quark mass tuned to its physical value, the light quark masses are varied in a range corresponding to pion masses $m_\pi \in [45{:}140]$ MeV. We substantially extended existing data sets for $H \in [1/160 : 1/27]$ and included new data sets with $H = 1/240$ for $T \in [143{:}147]$ MeV to probe the universal behavior. Results for M obtained for various H values and lattice sizes N_σ are shown in Fig. 2. For each vale of H the smallest and largest value of N_σ is given next to the data points. Results are shown for $T = 145.1$ MeV, which is our new preliminary estimate for T_c on $N_\tau = 8$ lattices. We will compare the numerical results with the finite-volume scaling function $f_{G\chi}(z, z_L)$ for the to 3-d $O(2)$ universality class, which is appropriate for calculations with staggered fermions at non-vanishing 'a'. In this scaling function, finite-size corrections are treated in terms of a Taylor series in z and z_L [5] :

$$f_{G\chi}(z, z_L) = f_{G\chi}(z, 0) + \sum_{m=m_l}^{m_u} \left(1 - \frac{1}{\delta} + \frac{m\nu}{\beta\delta}\right) a_{0m}\, z_L^m + \sum_{n=1}^{n_u} \sum_{m=m_l}^{m_u} \left(1 - \frac{1}{\delta} + \frac{n + m\nu}{\beta\delta}\right) a_{nm}\, z^n z_L^m \ . \quad (4)$$

While the second term of Eq.(4) projects the finite-volume corrections at $T = T_c$, the first term depicts the infinite-volume limit ($z_L = 0$) of $f_{G\chi}$ at different T which, at T_c is $f_{G\chi}(0, 0) = (1 - 1/\delta)$. For fixed T and H, we performed infinite-volume extrapolations using an ansatz inspired by Eq.(4), which however does not make any explicit use of universal properties :

$$f(T, H, L) = a_0(T, H) + a_3(T, H)/L^3 + a_4(T, H)/L^4 \ . \quad (5)$$

As apparent from Fig. 2, for $H > 1/80$, the $z_L = 0$ data points show significant deviations from an expected scaling behavior $M/H^{1/\delta} = h_0^{-1/\delta}$ (solid line) valid at T_c. The dashed line in Fig. 2 shows a 2% deviation from the infinite-volume extrapolated values. It is evident that the aspect ratio needed to reach such an accuracy increases from $N_\sigma/N_\tau \simeq 5$ at $H = 1/27$ to $N_\sigma/N_\tau \simeq 7$ at $H = 1/160$. One finds similar likewise implications for finite-volume effects in the right Fig. 3 plot, which shows the rescaled order parameter $M/H^{1/\delta}$ versus the finite-size scaling variable z_L for three different values of the temperature. Unlike Fig. 2, this figure makes explicit use of universal parameters in the 3-d, $O(2)$ universality class. It is evident that $H \geq 1/80$ data points deviate from the respective scaling curves. This is more severe for higher $H = 1/40$ with noticeable discrepancies even for large lattice volumes ($z_L \sim 0.3$). On the other hand, $H = 1/160, 1/240$ points show commendable agreement with expected universal finite-volume scaling even for $z_L \sim 0.6$, corresponding to smaller lattice volumes. Surely, additional $1/240$ data sets for larger volumes are still needed to arrive at firm conclusions. We also note that in a small temperature range around T_c *i.e.* 142.8 MeV $\leq T \leq$ 147.4 MeV, finite-volume effects are temperature independent to a large degree. This is observed in the left Fig. 3 plot, where the data points for $T = 142.8$ and 147.4 MeV, with $H \in [1/160 : 1/27]$, $N_\sigma \in [24 : 56]$ when shifted to $T = 145.1$ MeV, shows promising coincidence.

4 Summary and Outlook

We have demonstrated here that a systematic finite-volume analysis with controlled infinite-volume extrapolation for smaller quark mass values close to T_c indeed exhibit universal signatures with subdued finite-volume effects and sub-leading contributions. This facilitates a more precise estimate of T_c thereby providing a more stringent bound on QCD critical point [11], apart from quantitatively estimating the scaling regime and determining the underlying universality class. Nevertheless, more data close to the chiral critical point are still to be added besides, planning future simulations on $N_\tau = 12$ lattices. This is to obtain good enough precision for differentiating $U(2) \times U(2)$ and $O(4)$ universality classes in the continuum limit, which we hope would also potentially resolve the long-standing controversially discussed fate of $U(1)_A$ symmetry at the chiral phase transition temperature [12, 13].

Acknowledgements

This work is supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) Project No. 315477589-TRR 211. We thank the Bielefeld HPC.NRW and the NHR Center PC2 Paderborn teams for the computing times in the respective GPU clusters.

References

[1] H.T. Ding et al. (HotQCD), Phys. Rev. Lett. **123**, 062002 (2019), arXiv:1903.04801.
[2] J. Engels, F. Karsch, Phys. Rev. D **90**, 014501 (2014), arXiv:1402.5302.
[3] W. Unger, Ph.D. thesis, Bielefeld University (2010), https://pub.uni-bielefeld.de/record/2302381
[4] H.T. Ding et al., Phys. Rev. D **109**, 114516 (2024), arXiv:2403.09390.
[5] F. Karsch, M. Neumann, M. Sarkar, Phys. Rev. D **108**, 014505 (2023), arXiv:2304.01710.
[6] S. Mitra, F. Karsch, S. Sharma, PoS **LATTICE2024**, 187 (2025), arXiv:2411.15988.
[7] S. Mitra, F. Karsch, in *26th DAE-BRNS High Energy Physics Symposium 2024* (2025), arXiv:2501.13653
[8] F. Cuteri, O. Philipsen, A. Sciarra, JHEP **11**, 141 (2021), arXiv:2107.12739.
[9] L. Mazur, Ph.D. thesis, Bielefeld University (2021), https://doi.org/10.4119/unibi/2956493
[10] L. Mazur et al. (HotQCD), Comput. Phys. Commun. **300**, 109164 (2024), arXiv:2306.01098.
[11] F. Karsch, PoS **CORFU2018**, 163 (2019), arXiv:1905.03936.
[12] R.D. Pisarski, F. Wilczek, Phys. Rev. D **29**, 338 (1984).
[13] A. Pelissetto, E. Vicari, Phys. Rev. D **88**, 105018 (2013), arXiv:1309.5446.