

Factorial cumulants of proton multiplicity near a critical point using maximum entropy freeze-out prescription

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Abstract. We present the first application of the maximum-entropy freeze-out prescription to calculate factorial cumulants of proton multiplicities near the conjectured QCD critical point in thermal equilibrium. We use a parameterized class of possible EoS near QCD critical point, obtained via a mapping to the Gibbs free energy of the 3D Ising model. This equilibrium baseline highlights how factorial cumulants isolate critical fluctuations by subtracting trivial self-correlations, setting the stage for future out-of-equilibrium analyses. We identify the key nonuniversal aspects of the mapping to the Ising model that strongly control the characteristic properties, such as magnitude and location of the peaks of the factorial cumulants along the freeze-out curve.

1 Introduction

A central open question in QCD is whether a critical point lies on the boundary between the hadron resonance gas and the quark–gluon plasma. Event-by-event particle multiplicity fluctuations are among the most sensitive probes of such a critical point, with factorial cumulants being particularly useful since they subtract trivial self-correlations of a quantum gas. The STAR collaboration has published precision measurements of the factorial cumulants of proton multiplicity as a function of collision energy [2]. The data for the normalized second and third factorial cumulants of proton multiplicity show statistically significant deviations from the computed non-critical baselines [3]. To interpret such data and constrain the QCD equation of state near criticality, a framework is needed that connects QCD thermodynamics to the particle spectra measured after freeze-out.

In Ref. [1], we take a first step in this direction. Using a family of candidate QCD equations of state constructed via mapping to the universal 3D Ising model[4], we apply the maximum-entropy freeze-out prescription [5] to compute factorial cumulants of proton multiplicities in thermal equilibrium. This provides a controlled equilibrium baseline for future studies that incorporate the out-of-equilibrium dynamics expected near the critical point. Although finite-time effects are expected to become increasingly important near the critical point—making an equilibrium based calculation incomplete for heavy-ion collisions—it serves as a necessary prerequisite to more sophisticated out-of-equilibrium treatments.

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Since protons carry both baryon number and energy, the factorial cumulants of proton multiplicity depend on the cumulants of baryon density as well as those of energy density. The maximum-entropy prescription for freeze-out[5] maximizes the relative entropy of the hadron resonance gas (i.e. the entropy of fluctuations relative to those of an uncorrelated HRG) while ensuring that local hydrodynamic densities are matched across the kinetic and hydrodynamic descriptions at freeze-out on an event-by-event basis.

2 QCD Equation of State near the critical point from universality

The EoS of QCD at non-vanishing densities, and the phase diagram of QCD are less understood from first principles due to the notorious sign problem related to lattice QCD calculations. If the conjectured critical point of QCD exists, it would belong to the 3D Ising universality class. The universality implies that the EoS of QCD close to the critical point can be mapped to 3D Ising model via six non-universal (microscopic) parameters. The six non-universal parameters correspond to the values of the critical chemical potential, μ_c , the critical temperature, T_c , the angles that r and h axis make with the QCD μ axis on the $T - \mu$ phase diagram, denoted by α_1 and α_2 respectively and two scale factors ρ and w [4]. In this proceedings we make the following choices for two of the non-universal parameters: $\mu_c = 600$ MeV and $\alpha_2 = 0^\circ$. The values of the critical temperature, T_c and the slope of the first order line, α_1 then follow from the pseudo-critical boundary obtained from lattice calculations[6]. The critical point, the phase boundary and the freeze-out curves used for the analysis presented in this proceedings are shown in Fig. (1).

The hydrodynamic correlations in equilibrium can be directly calculated from the Equation of State. Ref.[5] presents a way to calculate the non-trivial correlations in a quantum gas of hadrons from the hydrodynamic correlations using maximum entropy freeze-out. For different values of the mapping parameters ρ and w , we study how the factorial cumulants of proton multiplicities vary along different freeze-out curves specified by

$$T_f(\mu_B) = T_{\text{crossover}}(\mu_B) - \Delta T_f \quad (1)$$

with $\Delta T_f = 4, 6$ and 9 MeV, where $T_{\text{crossover}}(\mu_B)$ is the Ising- r axis where $h = 0$.

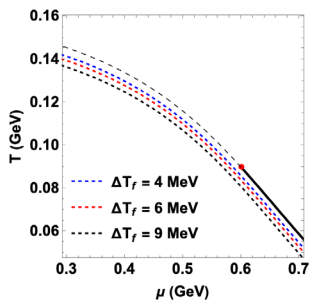


Figure 1. Three freezeout curves displaced downward relative to the crossover curve $\Delta T_f = 0$ (black) by $\Delta T_f = 4, 6$ and 9 MeV (dashed blue, red and black curves, respectively). An important non-equilibrium effect is that the fluctuations at freeze-out remember conditions at an earlier higher temperature, so it is possible that choosing a small ΔT_f in our calculation is a reasonable representation for freeze-out at a lower temperature in a calculation that includes non-equilibrium effects. For this reason ΔT_f in our calculation should be considered a model parameter to be fit to data.

3 Results on the phase diagram

For the presentation of the results here, we chose (w, ρ) values $(1, 1)$, $(5, 1)$ and $(5, 0.56)$ and ΔT_f values as specified in Fig. (1). The key observation is that while the position of the peak of the factorial cumulants of proton multiplicity along the freeze-out curve is controlled by the combination $\bar{\rho} = \rho w^{1-\frac{1}{\beta\delta}}$ [1, 7], the magnitude of the maximum value is more sensitive to w , and it goes as $w^{-1-1/\delta}$, where β and δ are the critical exponents for 3D Ising universality

class. ($\beta \approx 0.326$ and $\delta \approx 4.8$.) This can be seen in Figs. (2,3), where the second and third factorial cumulant of the proton multiplicity distribution, along the three freeze-out curves have been plotted. The $\rho = 0.56$ value for the rightmost panel in each of these plots have been chosen such that the $\bar{\rho}$ remains the same between the leftmost and rightmost plots while w changes. The magnitude of the maxima of the k^{th} factorial cumulant of proton multiplicity goes as $\Delta T_f^{1+1/\delta-k}$ [1].

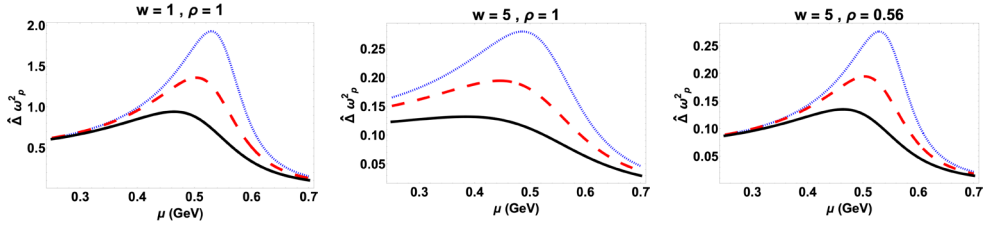


Figure 2. The second factorial cumulant of the proton multiplicity distribution, $\hat{\Delta}\omega_{2p}$, along the three freezeout curves from Fig. 1 characterized by Eq. (1) with $\Delta T_f = 4, 6$ and 9 MeV (blue dashed, red dashed and black dotted respectively). The panels show $\hat{\Delta}\omega_{2p}$ for various values of the nonuniversal mapping parameters w and ρ (specified on top of the figure), with $\mu_c = 600$ MeV, $T_c = 90$ MeV and $\alpha_2 = 0^\circ$. The peak height decreases as w increases, consistent with the scaling $w^{-6/5}$, while the location of the peak is controlled by the quantity $\bar{\rho} = \rho w^{2/5}$.

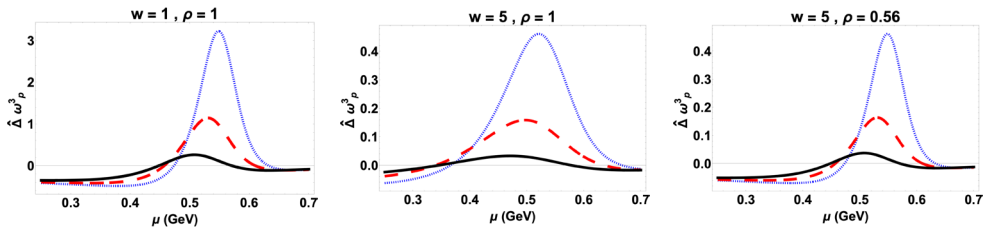


Figure 3. The third factorial cumulant of the proton multiplicity distribution, $\hat{\Delta}\omega_{3p}$, along the three freezeout curves from Fig. 1 characterized by Eq. (1) with $\Delta T_f = 4, 6$ and 9 MeV (blue dashed, red dashed and black dotted respectively).

Figs. (2,3), include only the contribution of direct protons, i.e those protons which existed at the time of freeze-out. In experiments, both direct protons as well as those coming from the decay products of heavier resonances contribute to the observed proton multiplicity distribution. We find that the inclusion of the child (decay) protons do change the estimates quantitatively, but donot change the qualitative trends. A comparison between the calculation including only direct protons and the one including both direct and child protons is shown for the choice of $\rho = w = 1$ and $\Delta T_f = 6$ MeV in Fig. (4).

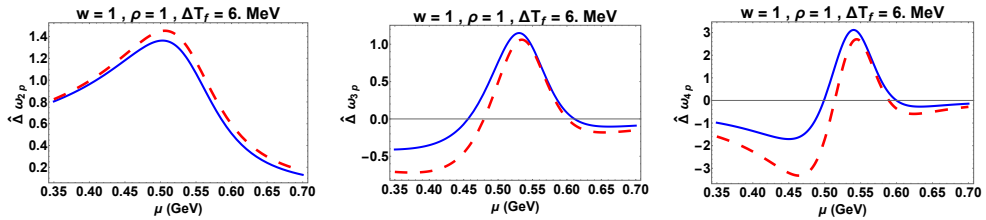


Figure 4. The second, third and fourth factorial cumulants of the proton multiplicity distribution, along the freezeout curve of Eq. (1) with $\Delta T_f = 6$ MeV. The red dashed curves show $\hat{\Delta}\omega_{4,p}$, which includes contributions of direct and child protons, whereas the blue solid curves include only the direct protons, as in Figs. 2 and 3.

4 Looking forward

Ref. [1], whose main results are summarized in this proceedings, presents the first application of the maximum-entropy freeze-out method to estimate experimentally accessible fluctuation observables—specifically, the factorial cumulants of proton multiplicity—for parametrically specified trial QCD equations of state. While this work assumes thermal equilibrium, more realistic calculations incorporating out-of-equilibrium effects remain an important next step. Looking ahead, a Bayesian comparison between experimental measurements of proton factorial cumulants and our theoretical baselines will provide quantitative constraints on the non-universal mapping parameters and, crucially, on the location of the QCD critical point.

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