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The contribution Hans Adolph Buchdahl f(R) gravity

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Abstract. Hans Adolph Buchdahl was born in Mainz, Germany, into a Jewish family. He was an Australian physicist, and he was born on 7 July 1919 and died on 7 January 2010. He worked on general relativity, thermodynamics, and optics. He was the founder of the modified theory of gravity of the $f(R)$ type, where $f(R)$ is a general function of the Ricci scalar, unlike Einstein's general theory of relativity $f(R) = R$. He proposed $f(R)$ gravity first in his paper in 1970 [2]. He set up the field equations of $f(R)$ gravity. Also, he is known for developing Buchdahl's theorem, which is a relation between the mass and radius in the static, spherically symmetric matter configurations. This paper will present his historical work on the creation of $f(R)$ gravity as the founder of $f(R)$ gravity, and also in one chapter it will be represented Buchdahl's theorem.

1 Introduction [1]

1.1 Short biography

Hans Adolf Buchdahl lived in Mainz, Germany. He was Jewish. He had an older brother, Gerd Buchdahl. His brother was a professor of philosophy. After Hitler became ruler of Germany in 1933, Gerd and Hans went to England. In London, he finished his B.Sc. and became an accomplice of the Royal College of Technological Know-How (ARCS) from Imperial college. In view of the fact that the second world battle had begun, many Jewish refugees have been sent to Australia. Hans and Gerd came to Australia in July 1940, on the boat of the HMT Dunera. Their ship was attacked twice via torpedoes for the duration of their two months journey to Australia. At the ship there have been many human beings. For instance, 2,036 Jewish refugees, 451 German and Italian prisoners of war, and the survivors of the Arandora celebrity. They arrived at Hay in New South Wales, and then they went to the Tatura center in Victoria in 1941. After he had become a mathematician, he got a job on the physics branch of the University of Tasmania in Hobart.

He worked in military research in optics for the duration of the war because he got the money for his work in optics. He used optical aberration of too excessive orders to mission the manner of imaging systems. His expressions are practiced in systems that we have on satellites nowadays. His equation in geometrical optics gives billions of dollars. With Pamela Wann, his wife from 1950, he had 3 children. He died in Adelaide, Australia, on 7 January 2010. He posted 160 studies papers and 5 books on his studies in general relativity, optics, and thermodynamics.

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1.2 University career

He became a research physicist in 1947.
Then he got a PhD from the University of Tasmania in 1949.
After that he elected in senior lecturer in 1952.
Then he got a 2nd PhD from Imperial University London.
He became a member of the Institute for Superior Study at Princeton from 1959 to 1960.
From 1963 he worked as a professor and leader of the department of Theoretical Physics within the college of technology at the Australian country wide university in Canberra.
He became Professor of Optics at the University of Rochester (New York) from 1967 to 1968.
He was an emeritus professor to the stop of his work.
And sooner or later, he became a pensioner in 1984–1985.

1.3 Awards

He got the subsequent awards:
Nuffield foundation Dominion Fellow in 1951.
Fellow of the Australian Academy of Technological in 1968.
The Academy's Thomas Ranken Lyle Medal in 1972.
An overseas Fellow of Churchill university, Cambridge in 1979.
An Australian country wide university (ANU) Fellow.
The Walter Burfitt Medal from the Royal Society of new South Wales in 1980.
The CEK Mees Medal from the Optical Society of the United States in 1993.
The AE Conrady Award of the international Society for Optical Engineering in 1997.

1.4 Buchdahl works

Except for optics, he labored on well-known relativity and classical thermodynamics. Heworked on education and the history of physics. He wrote two books on thermodynamics: The Standards of Classical Thermodynamics, posted in 1966, and the alternative, a sequence of twenty lectures on thermodynamics, published in 1975.

Hans published a key paper in general relativity. First, in his paper "General Relativistic Fluid Spheres," he presented Buchdahl's theorem. Buchdahl's theorem offers a modern boundary situation between the mass and the radius of a sphere, i.e., a sure, absolute restriction for all static fluid spheres whose density does not increase outside the sphere, the stress and density are greater than zero, and there exists a boundary wherein the pressure is infinite [9]. This boundary is the relation between mass and radius, which needs to be much less than 4/9, calculated by Schwarzschild for a simple case of constant density [1, 3, 9]. Buchdahl's model is greater realistic.

He corresponded with Einstein about his relativistic studies. He gave an extension of general relativity. In his paper [2], he proposed the f(R) version of gravity, i.e., he wrote a field equation for the f(R) model of gravity. He wrote books wherein the text was written with clarity but in a logical, concise, and simple way. He wrote lecture notes on general relativity as an e-book of Seventeen Easy Lectures about general relativity in 1981. His teaching had many informations and want for clarity and logical shape. He demanded excessive requirements, and he became a pretty professional person through his college students and personnel. Hans Adolph Buchdahl was not a public man as his contemporaries have been, but via sizeable use of the internet, we've discovered, for instance, that Hans is the founding father of the concept of f(R) gravity. Additionally, many scientists didn't take in their work Buchdahl theorem and didn't cite Buchdahl's works [2] because he became the founder of f(R) gravity. I think that

Hans needs to be among of the next scientists: I. Newton, J. Kepler, N. Copernicus, A. Einstein, HermanWeyl, and plenty of every other person who developed the celestial mechanics. He gave incredible contributions to theoretical physics, optics, and general relativity.

2 *f(R)* gravity [2]

Theory of general relativity is based on Einstain's equations which calculate from a variation of the density field Langragian $\Lambda = R + constant$ (R is Ricci scalar) and equations have the next shape: $P_{\mu\nu} = \pm kT_{\mu\nu}$, where k is constant, $P_{\mu\nu}$ is geometric tensor which depends on metric tensor $g_{\mu\nu}$ and their derivatives, and $T_{\mu\nu}$ is tensor of energy momentum which present the source of the gravitation field. Also tensor $T_{\mu\nu}$ satisfies

$$\nabla^{\mu} T_{\mu\nu} = 0, \tag{1}$$

where ∇^{μ} is covariant derivate. In Einstain theory of general relativity $P_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \lambda g_{\mu\nu}$. Buchdahl proposed to put the density Langragian is $\Lambda = f(R)$, general function of Ricci scalar R. He proposed following expressions for function $f(R) = \sum_0^n f_n R^n$ where f_n is constant which dimension is $(length)^{2n-2}$ and more realistic function $f(R) = \sqrt{\left(\left(R + \frac{1}{2l^2}\right)^2 + \frac{1}{2l^4}\right)}$, where l is fundamental length.

2.1 Derivation of Field Equation from Langrangian [2]

Lagrangian is given by the next expression:

$$L = \sqrt{-g} f(R), \tag{2}$$

where g is the determinant of metric tensor. Next equations are useful expressions. Variation of lagrangian by metric tensor $g_{\mu\nu}$ is:

$$\delta\left[\sqrt{-g}f\right] = \sqrt{-g}\left(\frac{df}{dR}\delta R + \frac{1}{2} g^{\alpha\beta} f \delta g_{\alpha\beta}\right) = \sqrt{-g}\left(\frac{df}{dR} g^{\alpha\beta}\delta R_{\alpha\beta} + (-\frac{1}{2} g_{\alpha\beta} f + \frac{df}{dR} R_{\alpha\beta})\delta g^{\alpha\beta}\right). \tag{3}$$

The expression for Ricci tensor is:

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\alpha,\nu} - \Gamma^{\alpha}_{\mu\nu,\alpha} + \Gamma^{\alpha}_{\varepsilon\nu}\Gamma^{\varepsilon}_{\mu\alpha} - \Gamma^{\varepsilon}_{\mu\nu}\Gamma^{\alpha}_{\varepsilon\alpha}. \tag{4}$$

Variation of Ricci tensor is:

$$\delta R_{\mu\nu} = \nabla_{\nu}\delta\Gamma^{\alpha}_{\mu\alpha} - \nabla_{\alpha}\delta\Gamma^{\alpha}_{\mu\nu}. \tag{5}$$

Variation of Christoffel symbol is:

$$\delta\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}\left(\nabla_{\nu}\delta g_{\beta\mu} + \nabla_{\mu}\delta g_{\beta\nu} - \nabla_{\beta}\delta g_{\mu\nu}\right). \tag{6}$$

where is ∇_{μ} is covariant derivate. For scalar $\nabla_{\mu} = \partial_{\mu} = \frac{\partial}{\partial x^{\mu}}$. The next two relations are the useful for determination of variation lagrangian:

$$g^{\mu\nu}\delta R_{\mu\nu} = \nabla_{\sigma}\left(g^{\mu\sigma}\delta\Gamma^{\alpha}_{\mu\alpha} - g^{\mu\nu}\delta\Gamma^{\alpha}_{\mu\nu}\right), \tag{7}$$

$$g^{\mu\nu}\delta R_{\mu\nu} = -g_{\mu\nu}\nabla_{\alpha}\nabla^{\alpha}\delta g^{\mu\nu} + \nabla_{\mu}\nabla_{\nu}\delta g^{\mu\nu}. \tag{8}$$

Finally equation of variation of langragian:

$$\frac{\delta\left(\sqrt{-g}f\right)}{\sqrt{-g}} = \frac{df}{dR}\left(-g_{\mu\nu}\nabla_\alpha\nabla^\alpha\delta g^{\mu\nu} + \nabla_\mu\nabla_\nu\delta g^{\mu\nu}\right) + (-\frac{1}{2}g_{\mu\nu}f + \frac{df}{dR}R_{\mu\nu})\delta g^{\mu\nu}. \tag{9}$$

Finally we get the geometric tensor that is proportional to the energy momentum tensor:

$$P_{\mu\nu} = -g_{\mu\nu}\nabla_\alpha\nabla^\alpha\frac{df}{dR} + \nabla_\mu\nabla_\nu\frac{df}{dR} - \frac{1}{2}g_{\mu\nu}f + \frac{df}{dR}R_{\mu\nu}. \tag{10}$$

And the equation from Buchdahl paper is:

$$\boldsymbol{g_{\mu\nu}\nabla_\alpha\nabla^\alpha\frac{df}{dR} - \nabla_\mu\nabla_\nu\frac{df}{dR} + \frac{1}{2}g_{\mu\nu}f - \frac{df}{dR}R_{\mu\nu} = kT_{\mu\nu}}, \tag{11}$$

$$\nabla_\mu\nabla_\nu\frac{df}{dR} = \frac{d^2f}{dR^2}\nabla_\mu\nabla_\nu R + \frac{d^3f}{dR^3}\nabla_\mu R\nabla_\nu R. \tag{12}$$

Final equation in Buchdahl paper from 1970 year is:

$$\boldsymbol{\left(-\nabla_\mu R\nabla_\nu R + g_{\mu\nu}\nabla_\alpha R\nabla^\alpha R\right)\frac{d^3f}{dR^3} + \left(-\nabla_\mu\nabla_\nu R + g_{\mu\nu}\nabla_\alpha\nabla^\alpha R\right)\frac{d^2f}{dR^2} - \frac{df}{dR}R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}f = kT_{\mu\nu}}. \tag{13}$$

2.2 Conclusion with examples

We assume metric for static spherical symmetric space [8]:

$$ds^2 = A(r)dt^2 - B(r)dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\varphi^2, \tag{14}$$

where: $g_{00} = A$, $g_{11} = -B$, $g_{22} = -r^2$, $g_{33} = -r^2\sin^2\theta$, $g^{\mu\mu} = \frac{1}{g_{\mu\mu}}$. The determinant of the metric tensor $g_{\mu\nu}$ is:

$$g = g_{00}g_{11}g_{22}g_{33} = -ABr^4\sin^2\theta. \tag{15}$$

Christoffel symbols $\Gamma^\alpha_{\varepsilon\nu}$ different from zero are:

$$\Gamma^1_{00} = \frac{1}{2B}\frac{dA}{dr}, \quad \Gamma^0_{10} = \Gamma^0_{01} = \frac{1}{2A}\frac{dA}{dr} \tag{16}$$

$$\Gamma^1_{11} = \frac{1}{2B}\frac{dB}{dr}, \tag{17}$$

$$\Gamma^1_{22} = -\frac{r}{B}, \quad \Gamma^1_{33} = -\frac{r}{B}\sin^2\theta, \tag{18}$$

$$\Gamma^2_{12} = \Gamma^3_{13} = \frac{1}{r}, \tag{19}$$

$$\Gamma^2_{33} = -\sin\theta\cos\theta, \tag{20}$$

$$\Gamma^3_{23} = \Gamma^3_{32} = \mathrm{ctg}\theta. \tag{21}$$

Ricci tensor $R_{\mu\nu}$ and Ricci scalar R are expressed by the next equations.
The 00 component is:

$$R_{00} = -\frac{1}{2B}\frac{d^2A}{dr^2} + \frac{1}{4B^2}\frac{dA}{dr}\frac{dB}{dr} + \frac{1}{4AB}\left(\frac{dA}{dr}\right)^2 - \frac{1}{rB}\frac{dA}{dr}. \tag{22}$$

The 11 component is:

$$R_{11} = \frac{1}{2A}\frac{d^2A}{dr^2} - \frac{1}{4A^2}\left(\frac{dA}{dr}\right)^2 - \frac{1}{4AB}\frac{dA}{dr}\frac{dB}{dr} - \frac{1}{rB}\frac{dB}{dr}. \tag{23}$$

The 22 component is:

$$R_{22} = \frac{1}{B} + r\frac{1}{2AB}\frac{dA}{dr} - r\frac{1}{2B^2}\frac{dB}{dr} - 1. \tag{24}$$

The 33 component is:

$$R_{33} = R_{22}\sin^2\theta. \tag{25}$$

The Ricci scalar is:

$$R = g^{\mu\nu}R_{\mu\nu} = -\frac{1}{AB}\frac{d^2A}{dr^2} + \frac{1}{2BA^2}\left(\frac{dA}{dr}\right)^2 + \frac{1}{2AB^2}\frac{dA}{dr}\frac{dB}{dr} + \frac{2}{r^2}\left(1 - \frac{1}{B}\right) + \frac{2}{rB^2}\frac{dB}{dr} - \frac{2}{rAB}\frac{dA}{dr}. \tag{26}$$

The field equations of unspecified $f(R)$ gravity are given in Buchdahl paper [2]:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\frac{f}{h} = \left(h_{;\mu\nu} - g_{\mu\nu}h^{\lambda}_{;\lambda}\right)\frac{1}{h}. \tag{27}$$

The trace equation is:

$$R = \frac{2f}{h} - \frac{3}{h}h^{\lambda}_{;\lambda}. \tag{28}$$

h is derivative of function $f(R)$:

$$h = \frac{df}{dR}. \tag{29}$$

The double covariant derivative of h is:

$$h_{;\mu\nu} = \frac{\partial^2 h}{\partial x^{\mu}\partial x^{\nu}} - \Gamma^{\lambda}_{\mu\nu}\frac{\partial h}{\partial x^{\lambda}}. \tag{30}$$

The covariant Dalamberian is:

$$h^{\lambda}_{;\lambda} = \frac{1}{\sqrt{-g}}\partial_{\mu}\left(\sqrt{-g}g^{\mu\nu}\partial_{\nu}h\right). \tag{31}$$

The covariant Dalamberian for the static spherical metric is:

$$h^{\lambda}_{;\lambda} = \left(-\frac{1}{2AB}\frac{dA}{dr} + \frac{1}{2B^2}\frac{dB}{dr} - \frac{2}{rB}\right)\frac{dh}{dr} - \frac{1}{B}\frac{\partial^2 h}{\partial r^2}, \tag{32}$$

where $h_{;\lambda}$ is covariant derivate.

2.2.1 THE FIRST EXAMPLE

In the first case we have the Einstein function

$$f(R) = R, \tag{33}$$

from which we get

$$h = 1. \tag{34}$$

The trace equation is :

$$Rh - 2f(R) - 3h^{\lambda}_{;\lambda} = 0, \tag{35}$$

$$R - 2R = 0, \tag{36}$$

$$R = 0. \tag{37}$$

We will take the bound limit:

$$AB = 1, \tag{38}$$

and from that we get for Ricci scalar the following equation:

$$R = -\frac{d^2A}{dr^2} - \frac{4}{r}\frac{dA}{dr} + \frac{2}{r^2}(1 - A). \tag{39}$$

Our trace equation has the next shape:

$$-\frac{d^2A}{dr^2} - \frac{4}{r}\frac{dA}{dr} + \frac{2}{r^2}(1 - A) = 0. \tag{40}$$

Finally, we get the Einstein solution for A:

$$A = 1 + \frac{r_s}{r} + \left(\frac{r_s}{r}\right)^2. \tag{41}$$

A is given by the next relation in Newton limit:

$$A = 1 - \frac{2V(r)}{c^2}, \tag{42}$$

where $V(r)$ is the gravitation potential, and c is velocity of light. In this case we get Einstein potential:

$$V(r) = -\frac{GM}{r}\left(1 + \frac{r_s}{r}\right). \tag{43}$$

2.2.2 THE SECOND EXAMPLE

In this example, we will take the next expression for the function $f(R)$:

$$f(R) = R^2. \tag{44}$$

We get the next expression for the h and his derivatives: $h = 2R$, $\frac{dh}{dR} = 2$, and $\frac{d^2h}{dR^2} = 0$.
The trace equation is :

$$Rh - 2f(R) - 3h^{\lambda}_{;\lambda} = 0, \tag{45}$$

$$2R - 2R - 3h^{\lambda}_{;\lambda} = 0, \tag{46}$$

$$h^{\lambda}_{;\lambda} = 0. \tag{47}$$

We take bound limit:

$$AB = 1, \tag{48}$$

$$h^{\lambda}_{;\lambda} = -\frac{dA}{dr} - \frac{2A}{r} = 0. \tag{49}$$

We get the solution for A:

$$A = \left(\frac{r_s}{r}\right)^2. \tag{50}$$

In this case we get for potential:

$$V(r) = \frac{c^2}{2}\left(1 - \left(\frac{r_s}{r}\right)^2\right), \tag{51}$$

where r_s is the Schwarzschilde radius.

3 Buchdahl Theorem [3]

Buchdahl proposed spherically symmetric general time invariant metric which take form [4–6]:

$$ds^2 = e^{\nu(r)}c^2dt^2 - e^{\lambda(r)}dr^2 - r^2d\theta^2 - r^2sin^2\theta d\varphi^2. \tag{52}$$

Matric elements of metric tensor are given:

$$g_{00} = e^{\nu(r)},\ g_{11} = -e^{\lambda(r)},\ \ g_{22} = -r^2,\ g_{33} = -r^2sin^2\theta. \tag{53}$$

General expression for Christoffel symbols are given:

$$\Gamma^{\alpha}_{\varepsilon\nu} = \frac{1}{2}g^{\alpha\sigma}(g_{\sigma\varepsilon,\nu} + g_{\sigma\nu,\varepsilon} - g_{\varepsilon\nu,\sigma}), \tag{54}$$

where

$$\Gamma^{\alpha}_{\mu\alpha,\nu} = \frac{\partial\Gamma^{\alpha}_{\mu\alpha}}{\partial x^{\nu}}, \tag{55}$$

and

$$g_{\mu\upsilon,\alpha} = \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}}. \tag{56}$$

Christoffel symbols $\Gamma^{\alpha}_{\varepsilon\nu}$ different from zero for Buchdahl proposed metric are [5]:

$$\Gamma^1_{00} = \frac{e^{\nu(r)-\lambda(r)}}{2}\frac{d\nu(r)}{dr}, \tag{57}$$

$$\Gamma^1_{11} = \frac{d\lambda(r)}{2dr}, \tag{58}$$

$$\Gamma^1_{22} = -re^{-\lambda(r)}, \tag{59}$$

$$\Gamma^1_{33} = -re^{-\lambda(r)}sin^2\theta, \tag{60}$$

$$\Gamma^1_{10} = \Gamma^1_{01} = \frac{d\nu}{2dr}, \tag{61}$$

$$\Gamma^2_{21} = \Gamma^3_{31} = \Gamma^2_{12} = \Gamma^3_{13} = \frac{1}{r}, \tag{62}$$

$$\Gamma^2_{33} = -sin\theta cos\theta, \tag{63}$$

$$\Gamma^3_{23} = \Gamma^3_{32} = ctg\theta. \tag{64}$$

General expression for Ricci tensor $R_{\mu\nu}$ is :

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\alpha,\nu} - \Gamma^{\alpha}_{\mu\nu,\alpha} + \Gamma^{\alpha}_{\varepsilon\nu}\Gamma^{\varepsilon}_{\mu\alpha} - \Gamma^{\varepsilon}_{\mu\nu}\Gamma^{\alpha}_{\varepsilon\alpha}. \tag{65}$$

The expressions for Ricci tensor R_{00} and R_{11} are :

$$R_{00} = e^{\nu(r)-\lambda(r)}\left(-\frac{d^2\nu}{2dr^2} - \frac{d\nu}{rdr} + \frac{d\nu}{4dr}\frac{d\lambda}{dr} - \frac{1}{4}\left(\frac{d\nu}{dr}\right)^2\right), \tag{66}$$

$$R_{11} = \frac{d^2\nu}{2dr^2} - \frac{d\lambda}{rdr} - \frac{d\nu}{4dr}\frac{d\lambda}{dr} + \frac{1}{4}\left(\frac{d\nu}{dr}\right)^2. \tag{67}$$

The equation of the Ricci scalar R is:

$$\mathrm{R} = e^{-\lambda(r)}\left(-\frac{d^2\nu}{dr^2} - \frac{2d\nu}{rdr} + \frac{d\nu}{2dr}\frac{d\lambda}{dr} - \frac{1}{2}\left(\frac{d\nu}{dr}\right)^2 + \frac{2d\lambda}{rdr} - \frac{2}{r^2}\right) + \frac{2}{r^2}, \tag{68}$$

Einstein equation for general relativity is:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \pm\frac{8\pi G}{c^4}T_{\mu\nu}. \tag{69}$$

To demand continuity of the stress-energy tensor:

$$\nabla_\mu T^\mu_\nu = 0, \tag{70}$$

where ∇_μ is covariant derivate, and the components of tensor energy-momentum are [4–6]:

$$T_{00} = \rho c^2 e^\nu, \tag{71}$$

$$T_{11} = T_{22} = T_{33} = Pe^\lambda. \tag{72}$$

There are three equations of field. The first equation is 00 component of the Ricci tensor [4–6]:

$$\frac{8\pi G}{c^2}\rho r^2 = 1 - e^{-\lambda} + r\frac{d\lambda}{dr}e^{-\lambda} \;\rightarrow\; e^{-\lambda} = 1 - \frac{2GM}{rc^2}. \tag{73}$$

The second equation is 11 component of the Ricci tensor [4–6]:

$$-\frac{8\pi G}{c^2}Pr^2 = 1 + e^{-\lambda}\left(-r\frac{d\nu}{dr} - 1\right). \tag{74}$$

We get the next shape of second equation when we substitute the solution of the first equation in the second equation [4–6]:

$$\frac{d\nu}{dr} = \frac{1}{r}\left[1 - \frac{2GM}{rc^2}\right]^{-1}\left[\frac{2GM}{rc^2} + \frac{8\pi G}{c^4}Pr^2\right]. \tag{75}$$

The relation of third equation is [4–6]:

$$\nabla_\mu T^\mu_\nu = 0, \tag{76}$$

i.e.

$$\frac{dP}{dr} = -\left(\frac{\rho c^2 + P}{2}\right)\frac{d\nu}{dr}. \tag{77}$$

So the final equation is Tolman-Oppenheimer-Volkoff equation or equation of hydrostatic equilibrium. This equation relates pressure and density of star.

$$\frac{dP}{dr} = -\frac{1}{r}\left(\frac{\rho c^2 + P}{2}\right)\left[1 - \frac{2GM}{rc^2}\right]^{-1}\left[\frac{2GM}{rc^2} + \frac{8\pi G}{c^4}Pr^2\right]. \tag{78}$$

We want to get closed system of equations, we need the equation of state. It is usually that the pressure depends on both density and entropy. We can often use the entropy is very small, and can be neglected. Then the equation of state takes the next form: $p = p(\rho)$. We will use simple model that is very realistic model of star and we assume that the fluid is incompressible: the density is constant out to the surface of the star. Also the mass inside of star dependes of radius and out of star is constant. The expressions for the density and the mass in this simple model are [7]:

Fot $r < R \Rightarrow \rho(r) = \rho \Rightarrow m(r) = \frac{4}{3}\pi r^3 \rho$.

Fot $r > R \Rightarrow \rho(r) = 0 \Rightarrow M = \frac{4}{3}\pi R^3 \rho$.

Integrated the Tolman-Oppenheimer-Volkoff equation we get the next shape for pressure in function of radius [7]:

$$P(r) = \rho c^2 \frac{R\sqrt{R - \frac{2GM}{c^2}} - \sqrt{R^3 - \frac{2GM}{c^2} r^2}}{\sqrt{R^3 - \frac{2GM}{c^2} r^2} - 3R\sqrt{R - \frac{2GM}{c^2}}}. \tag{79}$$

We defined the next expression:

$$D(r) = \sqrt{R^3 - \frac{2GM}{c^2} r^2} - 3R\sqrt{R - \frac{2GM}{c^2}}, \tag{80}$$

where $D(0)$ is:

$$D(0) = \sqrt{R^3} - 3R\sqrt{R - \frac{2GM}{c^2}}. \tag{81}$$

The pressure and density must be greater than zero, D(0) must less than zero.

$$D(0) < 0. \tag{82}$$

For the star of fixed radius R, the central pressure $p(0)$ will be greater than infinity if the maximal value of mass is :

$$M_{max} < \frac{4c^2 R}{9G}, \tag{83}$$

or the radius must be greater than:

$$R > \frac{9GM}{4c^2} = \frac{9}{8} r_s. \tag{84}$$

where r_s is the Schwarzschilde radius.

3.1 Conclusion for Buchdahl Theorem [3]

In the figure 1, we present the pressure inside the star Sun as a function of radius r. In the center of the star Sun the pressure is maximal, and the pressure decreases with increase radius r.

The central pressure diverges at the Buchdahl bound. The value R>9GM/4c^2 is an absolute lower limit for all static fluid spheres whose density does not increase outwards.

The conditions of Buchdahl's theorem are:

1) Spherical static symmetry.

2) Thermodynamic stability: density of matter is a non-increasing function of radial distance.

3)Radius-to-mass ratio limit is 9/4.

4) If an object exceeds the Buchdahl limit, it means that there exists a new type of object that is not covered by classical theory.

5) The central pressure p(0) will be infinity if the mass exceeds the Buchdahl limit.

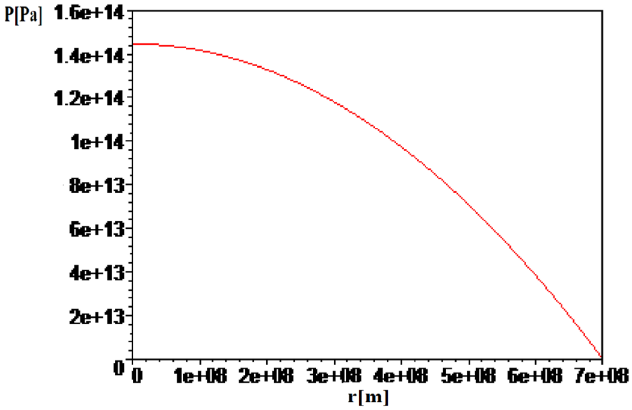

Figure 1. Central pressure P(Pa) of star Sun in the function of radius r(m). Radius of the Sun is: R=700000km, and mass density of the Sun is $\rho = 1500 kg/m^3$.

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