

From Present Observation to Future Uncertainty: Memory Dynamics in the Geometric Brownian Motion

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Abstract. This paper presents a theoretical and empirical analysis of a new property of stochastic processes - memory - that can be used to predict future values based on past observations. Memory is analyzed with respect to the way the information contained in one observation at time t propagates through the series of GBM-derived forecasts generated for future times $t + h$. More specifically, this analysis seeks to quantify how much the process carries forward (i.e., retains) the information from the current observation until the time of the forecast. Ultimately, the goal is to define the persistence of predictive information as the uncertainty associated with it increases with time. The analysis is based on the closed-form solution of the Geometric Brownian Motion and provides a mathematical relationship for the change in dispersion of forecasts and the consequent change in reliability of forecasts. The results suggest that, as a result of the properties of a non-mean-reverting diffusion with a variance that increases in direct proportion to time, the influence of the initial observation on subsequent observations quickly decays. We show evidence that the decline in memory occurs according to a logistic decay model and interpret this phenomenon as a law of memory decay. We use historical data on the S&P 500 index for empirical data validation. Once we tune the model's parameters, we create multi-horizon forecasts and develop associated prediction intervals. We then evaluate forecast accuracy through repeated validation at increasing horizon intervals of time, which allows us to quantitate the effective forecast life of the model. The findings support the theoretical memory pattern identified earlier and demonstrate the inherently short-term memory characteristic of Geometric Brownian Motion.

Keywords: Stochastic processes; GBM; Financial prediction; Logistic memory; Stochastic modeling.

1 Introduction

Stochastic modeling for financial time series is a key component in the world of modern quantitative finance [1]. The Geometric Brownian Motion (GBM) serves as one of the most commonly utilised stochastic processes for representing asset price movements. The GBM can describe the price of an asset through time continuously, experiencing exponential expansion and exhibiting proportional volatility throughout its life cycle. It is analytically tractable

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and has a closed-form solution, and as a result, many applications of GBM exist (asset pricing, risk analysis, (simulating) future market scenarios).

While there is a significant amount of literature dedicated toward exploring the mathematical properties of the GBM, the majority of the literature focuses on parameter estimation techniques, distributional characteristics and the statistical fit of the GBM to historical financial data. Although these areas are necessary for validating the empirical performance of the GBM, there is much more to stochastic processes than what has been investigated to date; specifically, there is also an implicit, or less obvious, and structural aspect to stochastic processes that is related to preserving and communicating information from previous observations when creating forecasts of future events [2].

The notion of predictive memory is presented in this research paper in relation specifically to the GBM. We describe a recall function that quantifies how much current observation contributes to the prediction of subsequent observations over increasing time horizons; this will be formalized through developing an explicit model of this relationship so that we can determine whether the GBM is behaving as a stochastic trajectory generating system or as a forecasting system; what role the persistence of past information plays on the forecasts generated by the GBM will be explored.

To test these claims empirically, we calibrate the GBM with historical data from the S&P 500. With the calibrated GBM, we construct discretized simulation equations to produce future price paths given the calculated parameters. From comparing these simulated price paths to actual market values, we can develop prediction intervals as well as determine the accuracy of our forecasts over many time horizons. The empirical framework developed in this way allows for an examination of the dynamics of predictive memory in a quantitative way.

This paper presents a way of estimating and interpreting the recall function of the geometric Brownian motion (GBM). This function represents how quickly information from the current observation will lose its usefulness as the forecast time continues to extend. Instead of limiting the analysis to traditional metrics such as short-term prediction error or volatility dynamics, the focus of the analysis will be on the premise of how long a model can effectively "remember" past observations when trying to generate future values.

The findings indicate that the predictions generated from a GBM model will be strongly influenced by the rapid increase in uncertainty associated with the forecasts made from that model, which will result in a continuing decrease in the effect of the information contained in the observation over time. By interpreting this effect in terms of the decay of memory, the article provides a much different perspective on stochastic financial modeling and presents a new means of interpreting diffusion-based models in terms of their predictive value and ability to generate value.

2 Literature Review

Financial economic models use probabilities to describe how uncertainty will affect asset prices over time. The Geometric Brownian Motion Model (GBM) dominates the marketplace because it is mathematically easy to use, has a log-normal distribution and has two very important features of assets: proportional volatility and exponential growth [3].

Numerous studies have been published related to the mathematics of the GBM model, the statistical characteristics of GBM model trajectories and the estimation of the parameters of GBM models based upon actual historical market price data. There are sufficient studies demonstrating the GBM model is one of the primary models that have been used in many financial modeling applications including: option pricing, portfolio valuation and risk man-

agement. The fact that the GBM model has a closed-form and has a Markovian structure makes it an extremely useful model for both theoretical researchers and practitioners.

Most current work on stochastic models has focused on improving the techniques used to estimate parameters [4], developing more sophisticated numerical techniques for running simulations, and extending the geometric Brownian motion (GBM) process to account for a variety of more complex phenomena that occur in the marketplace. These extensions often add features like jumps, stochastic volatility, or switching mechanisms, to capture empirical features of real-world asset price data, such as fat-tailed distributions of returns, clustering of volatility, and leverage. As well, many researchers have determined how accurately the GBM can predict the future of financial assets by looking at the GBM's forecast accuracy in sample and out of sample, or how well it can replicate the many statistical properties that are evident in the actual values of financial time series.

Despite all the research that has been done recently, there are still many aspects of stochastic models that have received very little attention. One such example is the implicit characteristic of the stochastic processes that govern the way in which the stochastic model retains and utilizes information from the past in order to generate forecasts for different timeframes. Even though the GBM is commonly conceived of as a memoryless Markov process in a true probabilistic manner, the GBM's process has an effective memory, due to how the forecasts are generated through recursive functions of observed data. However, this is an undervalued research area in contemporary diffusion models [5].

Many research papers have addressed how to modify the stochastic noise structure used in stochastic modeling to include memory-like behaviour (e.g., "stochastic processes with memory"). One example would be to use fractional Brownian motion as a different driving process to implement long- or short-range dependence in the behaviour of asset prices using Hurst's coefficient. This modification among other things adds to the temporal dependent structure of the stochastic model, however the modifications occur at the level of changing the stochastic driver rather than examining how much the informational persistence created by the classical GBM's forecasting mechanism affects an asset's return on investment [6].

Some other research studies have been conducted that compare the predictive ability of stochastic models against more traditional technical analysis indicators (i.e., moving averages, or momentum based oscillators) to determine which is more effective in forecasting future prices [7] [8]. Similarly, a growing number of studies have compared diffusion (diffusion-based) models to machine-learning types in regard to the trade-off between model interpretability, statistical efficiency, and forecasting accuracy.

Whereas earlier contributions have either focused on modelling volatility, calibrating statistics or explicitly modifying structure to add memory, the current research takes a different route [9] [10]. Instead of embedding memory into the stochastic differential equation itself we examine whether the standard GBM has predictive memories of past observations when forecasting. This type of memory can be quantified via recall functions derived from the dispersions and also reliability of multi-horizon forecasts.

By transitioning from parameter estimation and distribution fitting to assessing how long predictive information persists temporally, we propose a novel framework for evaluating stochastic models. Our intent is not to change a GBM's structure, but rather to identify and quantify the informational dynamics resulting directly from its diffusion characteristics when applied to empirical financial data. The perspective contributes an additional normative (temporal) and informational dimension to the study of stochastic processes in finance thus supplementing already established approaches.

3 Methodology

In order to rigorously define and quantify memory in a stochastic forecasting framework, we developed a simulation-based methodology that uses the Geometric Brownian Motion (GBM) as its foundation. This enables us to explore the Information Content (IC) of a current observable as it propagates through successive forecasts over extended (i.e., longer) prediction horizons.

We initially calibrated the parameters of the GBM to historical S&P500 data (2549 observations or 10 years of observations)[11]. We used the log-returns of the observational time series to obtain drift and volatility estimates, which emerged from this calibration. Using these parameters as foundation, we generated forecasts for one-step ahead, and computed prediction intervals at the 95% confidence level.

Next, we performed a series of recursive forecasting experiments that evaluated how predictive accuracy or performance, evolved as we iteratively extended our forecasts into the future beyond the one step ahead horizon. The experiments were structured in a manner that follows:

- **Step 1:** The GBM produces a forecast for the period after $j + 1$, based on actual observed market conditions at time j , and produces an accompanying prediction interval. If the actual observed market value at $j + 1$ lies within the prediction interval, then it will be assigned a "1" (true) indicator variable; otherwise, it will be assigned a "0" (false) indicator variable.
- **Step 2:** Using the projected value at time $j + 1$, the model calculates a prediction for time $j + 2$. Then, the observed value at time $j + 2$ is compared to the calculated prediction interval, and the indicator variable is updated.
- **Subsequent steps:** The Recursive process continues with more and more time periods added at each horizon. The forecast at horizon n uses the last simulated value as opposed to using any data that has been observed since that time, leading to a realistic multi-step forecasting experience.

At each point in time when predicting things out into the future, different standard statistical methods for evaluating how accurate are the predictions made by the model include precision, recall, and F1-score, etc., based on the degree to which they represent the actual value and are reliable; therefore, when you create models over the long term (i.e., if the forecasting period is extended), the width of the prediction interval also grows larger because, as time goes by, uncertainty builds up so that the quality of the model's predictions declines over time [12] [13].

The goal of studying this decline in predictive quality is not only to demonstrate that it occurs, but to describe its functional shape. The progressive loss of predictive quality is viewed as a reduction in memory of the first observation used for prediction as the dependence on simulated (i.e., not directly measured) rather than actual (i.e., directly measured) data grows over time. The relationship between predictive quality and forecasting lead time is termed the *memory law* of the process. In this sense, memory means the ability of the model to accurately predict outcomes (the same definition used earlier) while continuing to depend more and more on simulated versus actual observations.

Prior to discussing the empirical findings, we will provide a brief overview of the discrete-time simulation process associated with the GBM, estimation formulas to calibrate parameters, and how prediction intervals are constructed to assist with evaluation. These components of the analysis should provide the theoretical basis to interpret all subsequent results of the predictive memory analysis [14].

In general terms, a recursive relationship explains how the asset price at time t_{i+1} depends on the previous asset price, which is obtained as the sum of an expected (deterministic) growth component and a random (stochastic) shock component. The parameter σ is used to measure the size of random movements in the market, whereas the term γ represents drift adjusted for the discrete-time nature of the asset price series in the GBM.

To estimate the parameters of the process, we will use the fact that the logarithm of the ratio of price under the GBM is normally distributed. Let

$$y_{t_i} = \ln\left(\frac{X_{t_i}}{X_{t_{i-1}}}\right)$$

denote the sequence of observed log-returns computed from historical data. These returns form the basis for parameter estimation through Maximum Likelihood Estimation (MLE).

The sample mean of the log-returns is computed as

$$\hat{w} = \frac{1}{n} \sum_{i=1}^n y_{t_i} = \frac{\ln(X_n) - \ln(X_0)}{n},$$

while the corresponding sample variance is given by

$$\hat{\gamma} = \frac{1}{n} \sum_{i=1}^n (y_{t_i} - \hat{w})^2.$$

Using these empirical moments, the estimators of the GBM parameters can be expressed as

$$\hat{\sigma}^2 = \frac{\hat{\gamma}}{\Delta t}, \quad \hat{\mu} = \frac{1}{2} \hat{\sigma}^2 + \frac{\hat{w}}{\Delta t}.$$

After determining the drift and volatility coefficients from historic data, they can be used to simulate many different potential paths that GBM could take during a prediction horizon. The simulated paths are then used to create forecast intervals and to analyze the impact of increasing uncertainty on the validity of predictions. The analysis is key to evaluating the degree of persistence of predictive information and therefore to understanding the efficacy of memory in relation to GBM dynamics.

3.1 Prediction Interval Construction

To determine the credibility of the forecasts provided by the model, prediction intervals are built around each point forecast to represent the range of where observations occurring in the future will fall given some confidence level. The prediction intervals are based off of the statistical characteristics of the residuals created during model calibration.

If the regression residuals can be assumed to have a standard (approximately) normal distribution, then the prediction bounds can be computed by using the Student's t-distribution with $n-2$ degrees of freedom (n is equal to the number of observations used to calculate the parameter estimates). The standard error is a measure of the dispersion of the residuals and is calculated from the sum of the squared residuals (SSR).

Let s denote the estimated standard error, defined as

$$s = \sqrt{\frac{SSR}{n-2}}.$$

The prediction interval for a future observation y_{n+1} associated with an input value x_{n+1} is then given by:

$$y_{n+1} \in \left[\hat{y}_{n+1} \pm t_{n-2; 1-\alpha/2} \cdot s \sqrt{1 + \frac{1}{n} + \frac{(x_{n+1} - \bar{x})^2}{\sum_{i=1}^n (x_i - \bar{x})^2}} \right].$$

4 Results

The empirical data presented in this section was obtained using the previously described methodology and represents both intermediate computations and final indicators that were used to determine the predictive performance of Geometric Brownian Motion (GBM) when used with financial time series data.

To start, we will recall the estimated calibrated GBM parameter estimates, which were derived using historical observations of the S&P500 Index. These calibrated parameters were applied to the discrete time simulation equation used to create sequential forecasts. Lastly, the sequence of updates represents the stochastic relationship of two successive observations and captures the diffusion dynamics represented by the estimated calibrated drift and volatility coefficients.

Next, we produced 95% prediction limits for each forecasting horizon based on the simulation trajectories, which will help us evaluate the model's capacity to forecast the actual market with respect to the established prediction limits. Finally, we confirmed the accuracy of each forecast by determining whether the actual value is within or outside of the corresponding forecast limits.

A range of generic classification metrics were calculated, such as precision, recall, accuracy and F_1 -score, to evaluate the predictive performance across different time horizons; however, within the framework of this analysis, the definition of correct predictions is based strictly upon whether the seat value falls within the predicted interval. Therefore, since all predicted intervals are treated as positives in this binary context, subsequently we get a precision value of one for every horizon, therefore we anticipate using the dynamics of recall to form the basis upon which to assess the F_1 -score; thus we would believe that the F_1 -score is purely dependent upon recall and would therefore not provide any further independent information to the analysis.

For all subsequent analyses we shall therefore continue to use recall as the primary measure of performance; at each forecasting horizon recall provides a measure of the proportion of actual values contained within the prediction intervals generated recursively from the original observation; this measure will provide us with a dynamic profile of how quickly the predictive accuracy or reliability associated with the GBM deteriorates over time by computing successive iterations as the forecast time horizon increases.

The recall curve formulated can be considered as an empirical model of the longitudinal core (predictive memory) of a model's recall capability. It shows an inverse relationship between the amount of time that has passed from an initial observed value, and how far the value is ahead of time in determining future values, as measured on the recall curve (figure 1).

The loss of memory as depicted by the empirical data (on the recall curve) has a logistic-like pattern, including an initial gradual decline progressing to a fast declining rate, followed by long-term stabilization at low end (low probabilities) of the recall curve (longest forecasting periods). This behavior of lost memory is consistent with the properties of the geometric Brownian motion (GBM) theoretical model — namely, that the variance of the GBM increases linearly with time (thus producing variable lengths from time periods Etc.), which therefore will increase the likelihood that future observations will not be contained within the forecasted confidence range. [15].

In conclusion, it has been confirmed that there is a short time of effective predictive memory of Geometric Brownian Motion (GBM) as demonstrated using a recursive forecasting context. The shape of recall decay follows a logistic curve in terms of quantity, which describes how the information contained in one observed value becomes increasingly diluted

as forecasts are made into the future. This empirical pattern allows for the interpretation of recall as a quantifiable proxy for the memory component of the stochastic process.

4.1 Discretized Simulation Equation and Prediction Interval Estimation

The Geometric Brownian Motion was first fitted to the historical frequency data of the S&P 500 Index to develop the model parameters. Using the estimated drift and volatility from the earlier adjustments made to the discretised form of the process (as presented previously), the simulations create an equation that establishes a parameterisation for the relationship of two consecutive measurements of the variable.

$$X_{t_{i+1}} = X_{t_i} \exp(-0.000414 + 0.0112 Z_i), \quad \text{with } Z_i \sim \mathcal{N}(0, 1).$$

The above formula can recursively create synthetic index path over an increasing length of time for futures. When we run the Model, the simulated values are what we output, and they will then be compared to the corresponding, actual observed market value.

To determine how statistically reliable these predictions are, we develop a regression relationship between the simulated and observed values. The regression model allows us to construct a prediction interval of 95% (the interval within which we would expect future observations) for future observations as:

$$y_{n+1} \in \left[\hat{y}_{n+1} \pm 95.8 \sqrt{1 + \frac{(x_{n+1} - 3164.62)^2}{2434013330.3}} \right].$$

The prediction interval for the sample data provides an envelope of probability for the estimation of future index values at a high degree of confidence. In addition, it takes into account both the amount of variability in the residuals and where the predicted point is situated in relation to the sample.

The prediction intervals therefore define a point of reference for assessing the model's ability to forecast. Each time period is compared to the observed market value against its corresponding prediction interval. If the observed value is within the prediction interval, the forecast was successful and receives a score of 1; outside the prediction interval it will be marked as unsuccessful and receives a score of 0.

Through the continual assessment of the model to generate new points, we now have created a series of binary values that represents the predictive accuracy, or success or failure, of the model over various periods of time. From this series, standard measures for classification, such as precision, recall, accuracy, and the $F1$ score can be calculated. However, all predictions in this framework are treated as positive forecasts, resulting in a precision equal to 1 at each horizon. Therefore, the $F1$ score has the same numeric value as recall and adds nothing to our assessment of model performance.

Thus, recall remains the key measure of the predictive performance of forecasts. The evolution of recall as a function of the forecasting horizon enables the characterisation of gradual deterioration in the reliability of forecasts and, therefore, the quantification of the actual predictive memory of the Geometric Brownian Motion.

4.2 Logistic Memory of the Geometric Brownian Motion

To study the stability with respect to time of information for making predictions about the future using Geometric Brownian Motion as an example, analyses were performed on the evolution of recall scores over time, at each of the successively longer forecast horizons. Recall measures how many sets of observed values are still included in the prediction intervals

that were generated recursively from the original set of observations for each of the different forecast horizons. This series of recall scores gives us an empirical illustration of how rapidly predictive validity is lost via the accumulation of uncertainty over time.

The continuous and interpretable representation of this decay is created through the approximation of the previously calculated discrete recall values by a logistic function. The logistic function was chosen because of its ability to model non-linear transition dynamics as they relate to the initially very slow rate of change moving into and transferring to the extremely short rate of change followed by a very low rate of change (as compared to the starting point) going forward.

The logistic function that expresses the decay of memory for the Geometric Brownian Motion is:

$$\text{Recall}_{\text{GBM}}(x) = 14 + \frac{100}{1 + e^{1.1(x-2.3)}}.$$

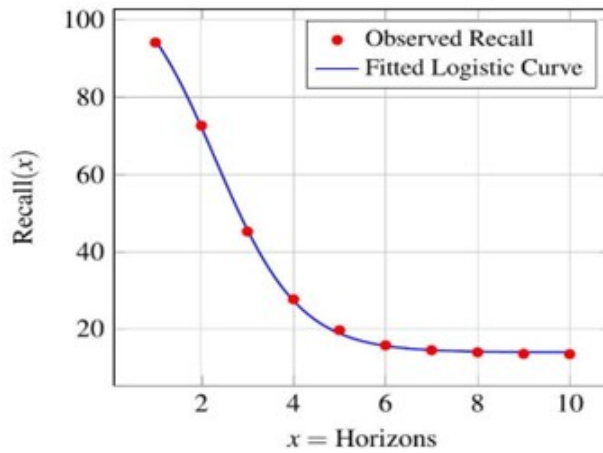


Figure 1. Logistic memory of GBM

5 Discussion

The study of stochastic finance is augmented by the introduction of predictive memory for evaluation of forecasting behaviour across time. Instead of restricting the evaluation of a model to its distributional assumptions or calibration accuracy, we evaluate how the informational characteristics of an observation can be continuously transformed and diminished as a model is recursively employed to generate multi-step forecasts. This perspective allows for a shift from the static statistical properties of a stochastic process to the temporal propagation of knowledge within stochastic dynamics.

The empirical evidence supports the notion that the evolution of recall as a function of the forecasting horizon is characterised by a logistic decay function. This functional form provides both a compact representation of the decay of predictive performance and interpretable parameters that summarise the effective length of time that useful information remains available in the forecasting process. The Geometric Brownian Motion Recall Analysis shows a systematic decline in recall accuracy as the forecast horizon increases. At the one-step-ahead forecast, or shortest horizon, recall is high due to the correlation between the last observed

value and the next value. The GBM prediction intervals are narrow enough to capture the next realized value with high probability.

Beyond this point, forecast accuracy steadily declines. Logistic regression represents an inflection point just after the second forecast step. This indicates a transition between good and bad forecasts, where observed values are now falling well outside of the predicted range at an increasing rate. Consequently, the model has little to no predictive capability.

The steepness of the logistic function indicates how sensitive the goal based model (GBM) is to uncertainty compounding. The reliance on simulated values, rather than newly observed data, for each forecast step means there is an accumulation of variability over time and therefore a rapid broadening of the plausible range of future trajectories for the GBM. This mechanism also weakens the statistical association between the first observation and forecasts for distant points in time, which reduces the amount of useful information available to the model from its initial observation.

At long horizons, recall stabilizes near 14%, which represents a baseline probability on forecasts with little utility as they do not provide meaningful predictive information regarding the actual future path of the asset. In this case, the width of prediction intervals is so large that their discriminative value is diminished, and the forecasts are also not of value for making decisions about the asset.

The GBM's empirical behavior supports the theoretical characteristics of GBM's diffusion process, driven by independent increments and exhibiting no mechanism of mean-reversion; therefore, GBM does not have an inherent structure for preserving information about the past states. Thus, the effect of the current observation diminishes exponentially as the forecast horizon grows longer, resulting in a very short effective predictive memory.

From a practical standpoint, this means that the GBM is best suited for probabilistic forecasting of all types over short time periods, where its simplicity in terms of analysis and clear distribution characteristics work to its advantage. However, due to its rapid decline in predictive memory, it has limited applicability as a reliable method for multi-step forecasting or long-term scenario analysis.

6 Conclusion

This research evaluated the Geometric Brownian Motion's (GBM) forecasting capabilities using a new concept called *predictive memory*. The analysis focused on how the recall of the current observation changes as the forecasting period increases to assess the loss of influence of past observations on future predictions made by the GBM. The empirical evidence shows recall's decay with time fits well to a logistic curve modelling function. Thus, the logistic curve gives a clear and concise account of how predictive performance degrades over time. With respect to the geometric brownian motion, the fitted logistic curve shows rapid degradation of recall with an increase in forecasting period by just a very small number of forecasts due to the GBM's rapid loss of predictive capability as the forecasting period expands. The behaviour of the GBM can be related directly to its underlying structure. Being a diffusion process with independent increments, the GBM does not retain memory of prior states over long periods of time, making it an ineffective long-term predictor. Thus, the sharp decline in recall observed from the empirical data corresponds to its limited predictive memory.

The way that looking mathematically at memory decay informs us about how effective stochastic forecasting may be over time, in that it allows for a quantitative summary of memory decay across the model parameters in the function that fits the data (i.e., slope and inflection point). Therefore, through the functional parameters, we are provided with a meaningful measure of the time over which the model continues to provide reliable information as well as how fast this reliability wanes. What this means in practice is that although the Geometric

Brownian Motion (GBM) continues to be a viable option for short-term probabilistic forecasting as well as for instances where an analytical formulation is needed, its rapid decline in predictive memory restricts its ability to meet the needs of multi-step forecasts with stability or long-range scenario projections.

Future research could explore if there are other types of stochastic processes with similar memory decay characteristics and evaluate this same logistic memory form in other asset classes and at other sampling frequencies to determine if it is a universal or invariant phenomenon.

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