New estimators in a partial linear model depending on an unbiased ridge regression estimator

Yousif A. Al-Khazraji [1*], ***Mustafa I. AIheety*** [2]

[1]Department of Mathematics, College of Education for Pure Sciences, University of Anbar, Anbar, Iraq.
[2]Department of Mathematics, College of Education for Pure Sciences, University of Anbar, Anbar, Iraq.

Abstract. This paper introduces two new estimators based on the philosophy of unbiased ridge regression estimation, where the parameters are part of a partial linear model suffering from multicollinearity. These proposed estimators are called the Difference-Based Unbiased Ridge Estimator $\hat{\beta}_{DB-URR}$ and the Difference-Based Modified Unbiased Ridge Estimator $\hat{\beta}_{DB-MUR}$ for the regression parameters β . The Mean Squared Error Matrix (MSEM) criterion is employed to compare the proposed estimators against the Difference-Based Ordinary Least Squares estimator $\hat{\beta}_{DB-OLS}$ and the Difference-Based Ordinary Ridge Estimator $\hat{\beta}_{DB-ORR}$. Finally, the performance of the new estimators is evaluated through a comprehensive simulation study and a numerical example.

Keywords: Partial Linear Model; Multicollinearity; Ridge Regression Estimator; Unbiased Ridge Regression Estimator; Difference-Based Estimator.

1. Introduction

In modern statistical and econometric analysis, the partial linear model has gained significant importance due to its ability to balance parametric precision with non-parametric flexibility. This model takes the general form:

$$y_i = \acute{x}_i\beta + f(t_i) + \epsilon_i, \qquad \text{i} = 1, \ldots\ldots, \text{n} \tag{1}$$

where $\acute{y}_i$ represents the i-th observation of the response variable, $\mathbf{x_i} = (x_{i1}, \ldots, x_{ip})'$ is a known p-dimensional vector of explanatory variables with $p \le n$, $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)'$ is a vector of unknown parameters, and $f(.)$ is an unknown smooth function. The non-parametric variables t_i are arranged and distributed so that $t_1 \le t_2 \le, \ldots, t_n$. The terms ϵ_i are independent random errors that follow a normal distribution with mean zero and variance σ^2, i.e., $\epsilon_i \sim N(0, \sigma^2)$.

* Corresponding author: you24u2005@uoanbar.edu.iq

We rewrite the model in equation (1) in matrix form as follows:

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{f}(\mathbf{t}) + \boldsymbol{\epsilon} \tag{2}$$

where $\mathbf{y} = (y_1, \dots, y_n)'$ is an $n \times 1$ vector of responses, $\mathbf{X} = (x_1, \dots, x_n\acute{)}$ is an $n \times p$ matrix , $\mathbf{f}(\mathbf{t}) = [(f(t_1), \dots, f(t_2)]'$ is an $n \times 1$ vector of the unknown function values, and $\boldsymbol{\epsilon} = (\epsilon_1, \dots, \epsilon_n)'$ is an $n \times 1$ vector of random errors. The partial linear model (2) comprises a linear component, $\mathbf{X}\boldsymbol{\beta}$, that is easily comprehended, and a nonlinear component, $\mathbf{f}(\mathbf{t})$, which reflects complicated interactions that are difficult to predict. This mixed character makes the model an effective tool for describing events that do not follow strictly linear assumptions. To accurately estimate the parametric coefficients $\boldsymbol{\beta}$ in the model, the effect of the non-parametric function $\mathbf{f}(\mathbf{t})$ must be eliminated. This is accomplished using the difference method suggested by Yatchew (1997, 2000)[1,2], which is an excellent tool for accomplishing this task. This approach works by sorting the data according to the non-parametric variable t and then calculating weighted differences between adjacent observations. This procedure essentially cancels out the term $\mathbf{f}(\mathbf{t})$, assuming that the function is locally linear (or nearly constant) over small intervals of t, converting the model into an approximate linear model that may be estimated using traditional techniques. Despite this transformation, the design matrix $\mathbf{X}$ associated with the parametric component is still subject to the multicollinearity problem among its independent variables. Multicollinearity is defined as the presence of nearly perfect linear dependence between the column vectors of the design matrix $\mathbf{X}$ in a linear model $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$. Here, $\mathbf{y}$ is an $n \times 1$ vector of observations, $\mathbf{X}$ is an $n \times p$ matrix of independent variables assumed to have full column rank p, $\boldsymbol{\beta}$ is a $p \times 1$ vector of unknown parameters, and ϵ is an error vector such that $E(\epsilon) = 0$, $E(\epsilon\epsilon') = \sigma^2 I_n$ (where $\boldsymbol{I_n}$ is the identity matrix of order n). Multicollinearity can result in large confidence intervals for individual parameters and erroneous signs for coefficients. The condition number of the matrix $\mathbf{X}$ indicates the presence of multicollinearity, but it does not show the structure of the linear dependence between its column vectors $x_1, x_2, \dots x_p$. Analyzing the eigenvalues of the matrix $\mathbf{X}'\mathbf{X}$ is the most effective method for identifying multicollinearity. This paper presents two new estimators, the Difference-Based Unbiased Ridge Estimator (DB-URR) and the Difference-Based Modified Unbiased Ridge Estimator (DB-MUR), and explores their application within the framework of the partial linear model after using the difference transformation. The remainder of the paper is organized as follows: Section 2 presents the related works. Section 3 defines the partial linear model and the differencing estimator and provides comprehensive mathematical formulae for the estimators used in the comparison: DB-ORR, DB-URR, and DB-MUR. Section 4 examines the theoretical performance of these estimators based on the Mean Squared Error (MSE) matrix criterion. Section 5 presents the simulation study and a numerical example using real-world data. Finally, Section 6 offers the key results and conclusions.

2. Related Works

Estimating the partial linear model often involves handling the non-parametric component effectively. For instance, Wang et al. (2007) [3] addressed this by using a unique class of difference sequences incorporating higher-order differences to enhance efficiency while estimating the linear component. However, their study primarily focused on efficiency and did not address the severe complications that arise when the explanatory variables are highly correlated. Indeed, multicollinearity causes the transformed design matrix $\acute{\mathbf{X}}\mathbf{X}$ to be ill-conditioned, leading to an inflation in the variance of ordinary least squares (OLS) estimators and a significant rise in the mean squared error (MSE). To account for the significant variability of OLS, numerous biased estimators have been devised. The most well-known is the Ordinary Ridge Regression in Linear Model (ORR) estimator, developed by Hoerl and Kennard (1970)[4], which decreases variance by adding a minor bias parameter (known as the Ridge parameter k) to the diagonal elements of the $\mathbf{\acute{X}X}$ matrix. In the context of the partial linear model, this Ridge logic was applied to the difference-transformed model; Tabakan and Akdeniz (2010)[5] proposed the difference-based Ridge estimator (DB-ORR) to decrease the huge variance in the difference-based least squares estimator (DB-OLS). However, a major limitation of the DB-ORR estimator is that it remains clearly biased, which can distort statistical inferences. To address the bias issue in standard linear models, Crouse et al. (1995)[6] proposed a unique theoretical framework, the Unbiased Ridge Estimator (URR), to bridge the gap caused by the Ridge estimator's explicit bias. This estimator tries to harness the variance-reduction capabilities of the Ridge estimator while retaining its unbiased nature by combining it with a priori information denoted by $\mathbf{J}$. Based on this concept, Batah 2009 [7] introduced the Modified Unbiased Ridge (MUR) estimator. MUR is a shrinkage estimator, derived from the URR in the same way that the (ORR) is derived from (OLS). The motivation behind this modification was that the URR, despite being unbiased, has a limitation:it still suffers from variance inflation under severe multicollinearity. Therefore, the MUR uses an additional shrinkage matrix **W** to improve the estimator's properties in terms of MSE and further reduce variance, even if this introduces some bias. Despite these advancements in standard linear models, there is a distinct gap in the literature regarding the application of URR and MUR estimators within the context of partial linear models. Therefore, to overcome the limitations of the biased DB-ORR and the high variance of DB-OLS, this paper bridges this gap by proposing the DB-URR and DB-MUR estimators, combining the strengths of difference-based transformations with advanced ridge regression methodologies.

3. Methodology

3.1 The model and differencing-based estimator.

To estimate β in model (2), eliminate the influence of the non-parametric function $\mathbf{f}(\mathbf{t})$. To do this, we employ the differencing approach proposed by Yatchew (1997, 2000,2003)[1, 2, 8]. This estimating technique is known as difference-based estimation. We have the weight vector $\mathbf{d} = (d_0, \dots, d_m)'$ of degree $m+1$, where m denotes the order of the differences and $d_0, \dots, d_m$ represent the weights of the differences that lower the variance in linear estimators, which means minimizing:

$$\min_{d_0,\dots,d_m} \delta = \sum_{i=1}^{m} \left(\sum_{j=0}^{m-i} d_j \quad d_{i+j} \right)^2 \tag{3}$$

Here, δ is called the difference series of order m^{th},and the weights must satisfy the following two conditions.

$$\sum_{j=0}^{m} d_j = 0 \quad , \quad \sum_{j=0}^{m} d_j^2 = 1 \tag{4}$$

The difference matrix $\mathbf{D}$ of dimension $(n-m) \times n$ whose elements satisfy conditions (4), is given by

$$\mathbf{D} = \begin{pmatrix} d_0 & d_1 & \dots & d_m & 0 & 0 & \dots & 0 \\ 0 & d_0 & d_1 & \dots & d_m & 0 & \dots & 0 \\ \dots & \dots & & & & & & \\ \dots & \dots & & & & & & \\ 0 & 0 & \dots & d_1 & \dots & d_m & 0 & 0 \\ 0 & 0 & \dots & d_0 & d_1 & \dots & d_m & 0 \\ 0 & 0 & \dots & 0 & d_0 & d_1 & \dots & d_m \end{pmatrix} \tag{5}$$

The difference matrix $\mathbf{D}$ performs a linear composition of adjacent observations after sorting the data in ascending order according to the variable t, which leads to the approximate removal of the function $\mathbf{f}(\mathbf{t})$ when applying the matrix $\mathbf{D}$ to both sides of the partial linear equation (2) as follows:

$$\mathbf{Dy} = \mathbf{DX}\boldsymbol{\beta} + \mathbf{Df}(\mathbf{t}) + \mathbf{D}\boldsymbol{\epsilon} \tag{6}$$

Since $\mathbf{Df}(\mathbf{t}) \approx \mathbf{0}$, we obtain the following transformed model

$$\mathbf{Dy} \cong \mathbf{D}\,\mathbf{X}\boldsymbol{\beta} + \mathbf{D}\boldsymbol{\epsilon} \tag{7}$$

We can rewrite equation (7) as:

$$\tilde{\mathbf{y}} \cong \tilde{\mathbf{X}}\boldsymbol{\beta} + \tilde{\boldsymbol{\epsilon}},$$

where $\tilde{\boldsymbol{y}} = \mathbf{Dy}$, $\tilde{\mathbf{X}} = \mathbf{DX}$, $\tilde{\boldsymbol{\epsilon}} = \mathbf{D}\boldsymbol{\epsilon}$.
So, we can see that $\tilde{\boldsymbol{\epsilon}}$ is a $n-m$ vector of distributed with a zero mean $E(\tilde{\boldsymbol{\epsilon}}) = 0$ and a variance $E(\tilde{\boldsymbol{\epsilon}}\tilde{\boldsymbol{\epsilon}}') = \sigma^2\,\mathbf{DD}'$. This transformation linearizes the model, allowing for the estimation of $\boldsymbol{\beta}$ using standard linear regression methods, notably the Ordinary Least Squares (OLS) approach. Using optimum weights, the OLS technique can provide the differencing estimator for the partial linear model suggested by Yatchew (1997)[1] for estimating $\boldsymbol{\beta}$, which takes the following form:

$$\hat{\boldsymbol{\beta}}_{DB-OLS} = \left(\tilde{\mathbf{X}}'\tilde{\mathbf{X}}\right)^{-1}\tilde{\mathbf{X}}'\tilde{\mathbf{y}} \tag{8}$$

Once $\boldsymbol{\beta}$ has been estimated, a non-parametric method can be used to estimate $\mathbf{f}(\mathbf{t})$. To calculate the variance, we use the estimator of σ^2 proposed by Eubank et al. (1998)[9] and Klippel, K., and Eubank, R. L. (2007) [10], as follows:

$$\hat{\sigma}^2 = \frac{\tilde{\mathbf{y}}'\,(\mathbf{I}-\mathbf{P})\tilde{\mathbf{y}}}{tr(\mathbf{D}'(\mathbf{I}-\mathbf{P})\mathbf{D})} \tag{9}$$

where $tr(.)$ denotes the trace of a square matrix and $\mathbf{P}$ is the projection matrix

$$\mathbf{P} = \tilde{\mathbf{X}}\left(\tilde{\mathbf{X}}'\tilde{\mathbf{X}}\right)^{-\mathbf{1}}\tilde{\mathbf{X}}' \tag{10}$$

The DB-OLS estimator has a very large variance in the presence of multicollinearity, which causes instability. To address this issue, Tabakan and Akdeniz (2010)[5] proposed the DB-ORR estimator, which is the most common biased estimator for dealing with multicollinearity, and its formula is given as:

$$\begin{aligned}\hat{\boldsymbol{\beta}}_{DB-ORR} &= \left(\tilde{\mathbf{X}}'\tilde{\mathbf{X}} + \mathbf{kI}\right)^{-1}\tilde{\mathbf{X}}'\tilde{\mathbf{y}} \\ &= \mathbf{C_k}\tilde{\mathbf{X}}'\tilde{\mathbf{y}}\end{aligned} \tag{11}$$

where $\mathbf{C_k} = \left(\tilde{\mathbf{X}}'\tilde{\mathbf{X}} + \mathbf{kI}\right)^{-\mathbf{1}}$ and $\mathbf{I_p}$ is the $\mathbf{p}\times\mathbf{p}$ identity matrix, and k is the ridge parameter, (also known as the bias coefficient). The condition $k \geq 0$, lowers variance by introducing a small amount of bias.

3.2 Difference-based unbiased ridge estimator and Modified Unbiased Ridge estimator

The proposed estimators use the conventional Ridge methodology combined with J, which represents prior knowledge about the location of the coefficient vector $\boldsymbol{\beta}$.The first suggested estimator, DB-URR, is a novel estimator that improves upon the performance of the standard ridge estimator DB-ORR in the partial linear model. This estimator was developed within a unique theoretical framework that uses the ridge parameter for minimizing variance while remaining unbiased. The formula is as follows:

$$\begin{aligned}\hat{\boldsymbol{\beta}}_{DB-URR} &= \left(\tilde{\mathbf{X}}'\tilde{\mathbf{X}} + \mathbf{kI}\right)^{-1}(\tilde{\mathbf{X}}'\tilde{\mathbf{y}} + \mathbf{kJ}) \\ &= \mathbf{C_k}(\tilde{\mathbf{X}}'\tilde{\mathbf{y}} + \mathbf{kJ})\end{aligned} \tag{12}$$

where $\mathbf{C_k} = \left(\tilde{\mathbf{X}}'\tilde{\mathbf{X}} + \mathbf{kI}\right)^{-\mathbf{1}}$ and $\mathbf{J}$ is a random vector representing prior information that follows a normal distribution with a mean $\boldsymbol{\beta}$ and a specified variance matrix $\left(\frac{\sigma^2}{k}\right)\mathbf{I}_p$.The second suggested estimator, DB-MUR, is a further refinement of the DB-URR estimator obtained by pre-multiplying it with a weighting matrix $\mathbf{W}$. It is defined by the formula:

$$\hat{\boldsymbol{\beta}}_{DB-MUR} = \mathbf{W}\;\hat{\boldsymbol{\beta}}_{DB-URR} \tag{13}$$

$$= \mathbf{W}\left(\widetilde{\mathbf{X}}'\widetilde{\mathbf{X}} + \mathbf{kI}\right)^{-\mathbf{1}}(\widetilde{\mathbf{X}}'\widetilde{\mathbf{y}} + \mathbf{kJ})$$

where $\mathbf{W} = \left[\mathbf{I_p} - \mathbf{k}\left(\widetilde{\mathbf{X}}'\widetilde{\mathbf{X}} + \mathbf{kI}\right)^{-\mathbf{1}}\right]$ is a shrinkage matrix. It seeks to lower the high variance in the DB-URR estimator while adding a small amount of bias. This adjustment enhances the shrinkage process by striking an optimal balance between bias and variance, resulting in better estimator qualities in terms of MSE.

4. Theoretical comparison in terms of MSE

Let us suppose $\mathbf{b}^*$ is an estimator for $\boldsymbol{\beta}$ in the linear model $\mathbf{y} = \mathbf{X\beta} + \boldsymbol{\epsilon}$. The mean square error Matrix (MSEM) of the estimator $\mathbf{b}^*$ is defined as follows.

$$\text{MSEM}(\mathbf{b}^*, \boldsymbol{\beta}) = \mathbf{E}[(\mathbf{b}^* - \boldsymbol{\beta})(\mathbf{b}^* - \boldsymbol{\beta})'] \tag{14}$$

If we designate the variance and covariance matrix of the estimator $\mathbf{b}^*$ by the symbol $\text{Var}(\mathbf{b}^*)$, the equation (14) is equivalent to

$$\text{MSE}(\mathbf{b}^*, \boldsymbol{\beta}) = \mathbf{V}(\mathbf{b}^*) + (\text{bias}(\mathbf{b}^*))\left(\text{bias}(\mathbf{b}^*)\right)'$$

where $\text{bias}(\mathbf{b}^*) = \text{E}(\mathbf{b}^*) - \beta$. The scalar Mean Squared Error (MSE) is provided by the relation. $\text{SMSE}(\mathbf{b}^*, \boldsymbol{\beta}) = \text{E}[(\mathbf{b}^* - \boldsymbol{\beta})(\mathbf{b}^* - \boldsymbol{\beta})'] = \text{tr}(\text{MSEM}(\mathbf{b}^*, \boldsymbol{\beta}))$

To make a comparison between any two estimators, we may need some of the lemmas below:

Lemma 4.1 Let $\widehat{\boldsymbol{\beta}}_{\mathbf{1}}$ and $\widehat{\boldsymbol{\beta}}_{\mathbf{2}}$ be the two estimators of $\boldsymbol{\beta}$. The estimator $\widehat{\boldsymbol{\beta}}_{\mathbf{2}}$ is said to be superior to $\widehat{\boldsymbol{\beta}}_{\mathbf{1}}$ in the sense of the MSEM criterion if [15]

$$\text{MSEM}\left(\widehat{\boldsymbol{\beta}}_{\mathbf{1}}\right) - \text{MSEM}(\widehat{\boldsymbol{\beta}}_{\mathbf{2}}) \geq 0$$

(i.e., the difference is a positive semi-definite matrix).

Lemma 4.2 (Farebrother, 1976) [16] Let $\mathbf{A}$ be a symmetric positive definite $m \times m$ matrix , $\mathbf{a}$ be an $m \times 1$ vector, and c be a positive number. Then the matrix $c\mathbf{A} - \mathbf{aa}'$ is a non-negative definite matrix if and only if $\mathbf{a}'\mathbf{A}^{-\mathbf{1}}\mathbf{a} \leq c$.

4.1 Difference-based unbiased ridge estimator and Modified Unbiased Ridge estimator

Tabakan and Akdeniz (2010)[5] defined the properties of the DB-ORR estimator as follows:

$$bias\left(\hat{\beta}_{DB-ORR}\right) = -k\mathbf{C_k}\beta \tag{15}$$

and

$$Var\left(\hat{\beta}_{DB-ORR}\right) = \sigma^2\mathbf{C_k}\mathbf{C}\,\mathbf{C_k} \tag{16}$$

where $\mathbf{C} = \left(\mathbf{D}'\widetilde{\mathbf{X}}\right)'\left(\mathbf{D}'\widetilde{\mathbf{X}}\right)$ and $\mathbf{C_k} = \left(\widetilde{\mathbf{X}}'\widetilde{\mathbf{X}} + \mathbf{kI}\right)^{-\mathbf{1}}$

To facilitate the SMSE comparison, we use the spectral decomposition (canonical form). Let $\mathbf{T}$ be an orthogonal matrix such that $\mathbf{TT}' = \mathbf{I_p}$, $\mathbf{T}\widetilde{\mathbf{X}}'\widetilde{\mathbf{X}}\mathbf{T}' = \boldsymbol{\Lambda} = \mathbf{diag}\left(\lambda_1, \dots, \lambda_p\right)$, and $\mathbf{T}'\mathbf{CT} = \Lambda = diag\left(c_1, \dots, c_p\right)$, where λ_i are the eigenvalues of $\widetilde{\mathbf{X}}'\widetilde{\mathbf{X}}$. We also define $\alpha = T'\beta$.The Scalar Mean Squared Error (SMSE) for DB-ORR can be expressed in the canonical form as:

$$\begin{aligned}\text{SMSE}\left(\widehat{\boldsymbol{\beta}}_{DB-ORR}\right) &= tr(Var\left(\widehat{\boldsymbol{\beta}}_{DB-ORR}\right)) + bias\left(\widehat{\boldsymbol{\beta}}_{\boldsymbol{DB-ORR}}\right)\boldsymbol{bias}\left(\widehat{\boldsymbol{\beta}}_{\boldsymbol{DB-ORR}}\right)' \\ &= tr(\sigma^2\mathbf{C_k}\mathbf{C}\mathbf{C_k}) + k^2\mathbf{C_k}\,\boldsymbol{\beta}\boldsymbol{\beta}'\mathbf{C_k} \\ &= \sigma^2\sum_{i=1}^{p}\frac{c_i}{(\lambda_i + k)^2} + k^2\sum_{i=1}^{p}\frac{\alpha_i^2}{(\lambda_i + k)^2}\end{aligned} \tag{17}$$

Now, we will find out the properties of the new estimator DB-URR

Using equation (12), we obtain

$$E\left(\widehat{\boldsymbol{\beta}}_{DB-URR}\right) = \boldsymbol{\beta} \tag{\mathbf{18}}$$

It is an unbiased estimator, meaning the bias is zero; thus the estimated mean squared error matrix (MSEM) of the (DB-URR) is exactly equal to its variance matrix.

$$\mathbf{MSEM}\left(\widehat{\boldsymbol{\beta}}_{DB-URR}\right) = Var\left(\widehat{\boldsymbol{\beta}}_{DB-URR}\right) = \sigma^2\boldsymbol{C_k}[\boldsymbol{C} + k\boldsymbol{I}]\boldsymbol{C_k} \tag{19}$$

The SMSE of the DB-URR estimator is exactly equal to the trace of its variance matrix, which can be written in the canonical form as:

$$\mathrm{SMSE}\big(\widehat{\boldsymbol{\beta}}_{DB-URR}\big) = tr(\sigma^2 \mathbf{C_k}[\mathbf{C} + \mathrm{k}\mathbf{I}]\mathbf{C_k}) = \sigma^2 \sum_{i=1}^{p} \frac{c_i + k}{(\lambda_i + k)^2} \tag{20}$$

Now, we investigate the difference between the SMSEs of the two estimators to establish superiority.

$$\begin{aligned}\Delta = {} & \mathrm{SMSE}\big(\widehat{\boldsymbol{\beta}}_{\mathrm{DB-ORR}}\big) - \mathrm{SMSE}\big(\widehat{\boldsymbol{\beta}}_{\mathrm{DB-URR}}\big) \\ = {} & \sum_{i=1}^{p} \frac{\sigma^2 c_i + k^2\alpha_i^2 - \sigma^2(c_i + k)}{(\lambda_i + k)^2} \\ = {} & \sum_{i=1}^{p} \frac{k}{(\lambda_i + k)^2}\left[\mathrm{k}\alpha_i^2 - \sigma^2\right]\end{aligned} \tag{21}$$

For the proposed estimator $\widehat{\boldsymbol{\beta}}_{\mathrm{DB-URR}}$ to be superior to $\widehat{\boldsymbol{\beta}}_{\mathrm{DB-ORR}}$, the difference Δ must be non-negative (i.e., $\Delta \geq 0$). This yields the following theorem:

Theorem 4.1 The proposed estimator $\hat{\beta}_{DB-URR}$ is superior to the estimator $\hat{\beta}_{DB-ORR}$ in terms of the Scalar Mean Squared Error (SMSE) if and only if

$$k\alpha_i^2 > \sigma^2 \quad , \quad \forall \mathrm{i} = 1, \dots, \mathrm{p} \tag{22}$$

4.2 Comparing the $\widehat{\boldsymbol{\beta}}_{DB-MUR}$estimator with the $\widehat{\boldsymbol{\beta}}_{DB-ORR}$estimator

Now, we investigate the properties of the second proposed estimator $\widehat{\boldsymbol{\beta}}_{DB-MUR}$Using equation (13), we obtain the estimator's expectation

$$E\big(\hat{\beta}_{DB-MUR}\big) = E\big(\mathbf{W}\ \widehat{\boldsymbol{\beta}}_{DB-URR}\big) = \mathbf{W}\boldsymbol{\beta} \tag{23}$$

and its bias is :

$$bias\big(\hat{\beta}_{DB-MUR}\big) = -k\mathbf{C_k}\boldsymbol{\beta} \tag{24}$$

We note that the estimated bias of $\widehat{\boldsymbol{\beta}}_{DB-MUR}$ is exactly equal to the estimated bias $\hat{\beta}_{DB-ORR}$The variance matrix of the DB-MUR estimator is known

$$Var\big(\hat{\beta}_{DB-MUR}\big) = \sigma^2 \mathbf{W}\mathbf{C_k}[\mathbf{C} + \mathrm{k}\mathbf{I}]\mathbf{C_k}\mathbf{W}' \tag{25}$$

Therefore, the mean squared error matrix (MSEM) is formulated as**:**

$$\mathrm{MSEM}\big(\widehat{\boldsymbol{\beta}}_{DB-MUR}\big) = Var\big(\widehat{\boldsymbol{\beta}}_{DB-MUR}\big) + \Big(bias\big(\widehat{\boldsymbol{\beta}}_{DB-MUR}\big)\Big)\Big(bias\big(\widehat{\boldsymbol{\beta}}_{DB-MUR}\big)\Big)'$$

Let $\mathbf{b_1} = bias\big(\widehat{\boldsymbol{\beta}}_{DB-MUR}\big) = -k\mathbf{C_k}\boldsymbol{\beta}$

$$\begin{aligned}&= Var\big(\widehat{\boldsymbol{\beta}}_{DB-MUR}\big) + (\mathbf{b_1})(\mathbf{b_1})' \\ &= \sigma^2 \mathbf{W}\mathbf{C_k}[\mathbf{C} + k\mathbf{I}]\mathbf{C_k}\mathbf{W}' + (-k\mathbf{C_k}\boldsymbol{\beta})(-k\mathbf{C_k}\boldsymbol{\beta})'\end{aligned}$$

$$MSE\big(\hat{\beta}_{DB-MUR}\big) = \sigma^2 \mathbf{W}\mathbf{C_k}[\mathbf{C} + k\mathbf{I}]\mathbf{C_k}\mathbf{W}' + k^2 \mathbf{C_k}\boldsymbol{\beta}\boldsymbol{\beta}'\mathbf{C_k} \tag{26}$$

Now let's look at the difference between the Mean Squared Error (MSEM) matrix

$$\begin{aligned}\Delta = {} & MSEM\big(\widehat{\boldsymbol{\beta}}_{DB-ORR}\big) - MSEM\big(\widehat{\boldsymbol{\beta}}_{DB-MUR}\big) \\ = {} & \big[Var\big(\widehat{\boldsymbol{\beta}}_{DB-ORR}\big) + (\mathbf{b})(\mathbf{b})'\big] - \big[Var\big(\widehat{\boldsymbol{\beta}}_{DB-MUR}\big) + (\mathbf{b_1})(\mathbf{b_1})'\big] \\ = {} & \big[Var\big(\widehat{\boldsymbol{\beta}}_{DB-ORR}\big) - Var\big(\widehat{\boldsymbol{\beta}}_{DB-MUR}\big)\big] + [\,(\mathbf{b})(\mathbf{b})' - (\mathbf{b_1})(\mathbf{b_1})'\,]\end{aligned}$$

Since $\mathbf{b} = \mathbf{b_1} = -k\mathbf{C_k}\boldsymbol{\beta}$, the squared bias matrices cancel each other out. Thus, the MSEM difference reduces exclusively to the difference in their variance matrices:

$$\begin{aligned}\Delta = {} & Var\big(\widehat{\boldsymbol{\beta}}_{DB-ORR}\big) - Var\big(\widehat{\boldsymbol{\beta}}_{DB-MUR}\big) \\ = {} & \sigma^2 \mathbf{C_k}\mathbf{C}\mathbf{C_k} - \sigma^2 \mathbf{W}\mathbf{C_k}[\mathbf{C} + k\mathbf{I}]\mathbf{C_k}\mathbf{W}'\end{aligned} \tag{27}$$

To facilitate comparison and verify the positivity definiteness condition, we use spectral decomposition to transform these matrices into their canonical form.Equation (27) can be written using the canonical form as.

$$diag\left\{\frac{\sigma^2 k}{(\lambda_i + k)^4}\quad [2c_i\,\lambda_i + c_i\,k - \lambda_i^2\,]\right\}_{i=1}^{p}$$

This results in the matrix Δ being a positively definite (p.d.) matrix if and only if all its diagonal elements are non-negative. This simplifies to:

$$k > \frac{\lambda_i^2}{c_i} - 2\lambda_i$$

Theorem 4.2 The estimator of $\hat{\beta}_{DB-MUR}$ is superior than the estimator of $\hat{\beta}_{DB-ORR}$ in terms of the Mean Squared Error Matrix (MSEM) if and only if

$$k > \frac{\lambda_i^2}{c_i} - 2\lambda_i \qquad \forall i = 1, \dots, p \tag{28}$$

4.3 Comparing the $\widehat{\boldsymbol{\beta}}_{DB-MUR}$ estimator with the $\widehat{\boldsymbol{\beta}}_{DB-URR}$ estimator

The difference between the variance matrices can be written from equations (19) and (25) as follows:

$$\begin{aligned} \boldsymbol{V} &= Var\left(\widehat{\boldsymbol{\beta}}_{DB-URR}\right) - Var\left(\widehat{\boldsymbol{\beta}}_{DB-MUR}\right) \\ &= \sigma^2 \mathbf{C_k}[\mathbf{C} + \mathrm{k}\mathbf{I}]\mathbf{C_k} - \sigma^2 \mathbf{W}\mathbf{C_k}[\mathbf{C} + \mathrm{k}\mathbf{I}]\mathbf{C_k}\mathbf{W}' \end{aligned} \tag{29}$$

Since $\mathbf{W} = \mathbf{I} - \mathrm{k}\,\mathbf{C_k}$ is a shrinking matrix, its eigenvalues are strictly bounded between 0 and 1, making $\mathbf{V}$ a strictly positive definite matrix. Now let's look at the difference between the Mean Squared Error Matrix (MSEM):

$$\begin{aligned} \Delta = & \ \mathrm{MSEM}\left(\widehat{\boldsymbol{\beta}}_{DB-URR}\right) - \mathrm{MSEM}\left(\widehat{\boldsymbol{\beta}}_{DB-MUR}\right) \\ &= \left[Var\left(\widehat{\boldsymbol{\beta}}_{DB-URR}\right)\right] - \left[\, Var\left(\widehat{\boldsymbol{\beta}}_{DB-MUR}\right) + (\mathbf{b_1})(\mathbf{b_1})'\right] \\ &= \sigma^2 \mathbf{C_k}[\mathbf{C} + \mathrm{k}\mathbf{I}]\mathbf{C_k} - \boldsymbol{\sigma^2} \mathbf{W}\mathbf{C_k}[\mathbf{C} + \mathrm{k}\mathbf{I}]\mathbf{C_k}\mathbf{W}' - \ \mathbf{k^2}\mathbf{C_k}\boldsymbol{\beta}\boldsymbol{\beta}'\mathbf{C_k} \\ &= k^2 \boldsymbol{C_k}\left[\ \sigma^2\left[\frac{1}{k^2}(\mathbf{C} + \mathrm{k}\mathbf{I}) - \frac{1}{k^2}\mathbf{W}(\mathbf{C} + k\mathbf{I})\mathbf{W}'\right] - \boldsymbol{\beta}\boldsymbol{\beta}'\right]\mathbf{C_k} \\ &= k^2 \mathbf{C_k}\left[\ \sigma^2 \mathbf{A_1} - \boldsymbol{\beta}\boldsymbol{\beta}'\right]\mathbf{C_k} \end{aligned}$$

where $\mathbf{A} = \frac{1}{k^2}[(\mathbf{C} + \mathrm{k}\mathbf{I}) - \mathbf{W}(\mathbf{C} + \mathrm{k}\mathbf{I})\mathbf{W}']$. since $\boldsymbol{A}$ is a positive definite matrix and $\sigma^2 > 0$. (see, e.g.,Trenkler and Toutenburg 1990 [11]; Rao and Toutenburg 1995 [12]; Groß 2003 [13]).By Lemma 4.2 (Farebrother, 1976)[16], we obtain $\sigma^2\mathbf{A} - \boldsymbol{\beta}\boldsymbol{\beta}'$ and Δ is nonnegative definite matrix if and only if

$$\boldsymbol{\beta}'\mathbf{A^{-1}}\boldsymbol{\beta} \le \sigma^2$$

Theorem 4.3 The estimator of $\hat{\beta}_{DB-MUR}$ is superior than the estimator of $\hat{\beta}_{DB-URR}$ in terms of the Mean Squared Error Matrix (MSEM) if and only if

$$\boldsymbol{\beta}'\mathbf{A^{-1}}\boldsymbol{\beta} \le \sigma^2 \quad , \quad \forall i = 1, \dots, p \tag{30}$$

5. Simulation Study and numerical example

5.1 Simulation Study

In this section, the Estimated Mean Squared Error (EMSE) is employed as a criterion to compare the estimators and determine the optimal performance. The effects of sample size and correlation strength were considered, alongside the selection of various values for the ridge parameter (k). To meet the requirements for robust testing three different sample sizes were chosen: $n = 50{,}100{,}200$. To examine the estimators' consistency, small, medium, and large samples were used. Additionally, four levels of correlation were investigated: $\rho = 0.8{,}0.9{,}0.95{,}0.99$. These numbers were intentionally chosen to represent high to severe levels of multicollinearity, which is the primary difficulty that these estimators are intended to address. with the number of variables set to $(p = 5)$. The standard deviation of the error terms was set to three levels: σ = 0.1,1,5.Third-order difference coefficients $(m = 3)$were utilized, with values $d_0 = 0.8582\,, d_1 = -0.3832\,, d_2 = -0.2809, d_3 = -0.1942$. Optimal difference coefficients for $1 \le m \le 10$ can be found in Yatchew (2003)[8].The explanatory (independent) variables were generated using a Monte Carlo simulation method based on the equation proposed by McDonald and Galarneau (1975) [14] :

$$x_{ij} = (1-\rho^2)^{1/2} w_{ij} + \rho w_{i(p+1)}\,, \quad i = 1,2,\dots,n\ ; j = 1,2,\dots,p \tag{31}$$

where w_{ij} are independent pseudo-random numbers following the standard normal distribution, and ρ^2represents the theoretical correlation between any two explanatory variables. The model is generated according to the following equation:

$$y_i = \sum_{j=1}^{5} x_{ij}\beta_j + f(t_i) + \varepsilon_i \tag{32}$$

where $\beta = (1.5,3,2,-5,4)'$, and $\epsilon_i\ \sim N(0,\sigma^2)$. The non-parametric component is defined by the Doppler function $f(t_i) = \sqrt{t_i(1-t_i)}\sin(2.1\pi/(t_i+0.05))$.where $t_i = (i-0.5)/n$ for $i = 1,2,\dots,n$ The experiment was replicated 2000 times by generating new error terms while the matrix **X**, the vector $\boldsymbol{\beta}$, and the function **f** remained fixed. This large number of replications ensures that the variations in EMSE are strictly due to the random component and guarantees statistically stable results. After generating 2000 samples, the estimated Mean Squared Error (EMSE) was calculated for each estimator. The EMSE is defined by the formula:

$$EMSE(\widehat{\boldsymbol{\beta}}) = \frac{1}{2000}\sum_{j}^{2000}\sum_{i}^{5}\left(\widehat{\boldsymbol{\beta}}_{ij} - \boldsymbol{\beta}_i\right)^2 \tag{33}$$

where $\widehat{\boldsymbol{\beta}}_{ij}$ denotes the estimate of the i parameter in the j replication, and $\boldsymbol{\beta}_i$represents the true parameter values. Following the approach suggested by Crouse et al, the prior information vector $\mathbf{J}$ was determined empirically. To maintain consistency with the difference-based structure of our model, we utilized the Difference-Based Ordinary Least Squares DBOLS estimates. Specifically, we calculated the arithmetic mean of the DBOLS coefficients and assigned this average value to all elements of the prior vector. Mathematically, $\mathbf{J}$ is defined as: $\mathbf{J} = \left(\frac{1}{p}\sum_{i=1}^{p}\beta_{i,_{DBOLS}}\right) \times 1_{P\times 1}$, where $1_{P\times 1}$is a vector of ones. It is also worth noting that when $k = 0$, the estimators become equivalent, such that $\widehat{\boldsymbol{\beta}}_{DB-OLS} = \widehat{\boldsymbol{\beta}}_{DB-ORR} = \widehat{\boldsymbol{\beta}}_{DB-URR} = \widehat{\boldsymbol{\beta}}_{DB-MUR}$The simulation was conducted using the MATLAB R2025b software package.

https://doi.org/10.1051/epjconf/202636902008

n=50 σ=0.1																					
ρ=0.8																					
k	0	0.001	0.005	0.006	0.007	0.009	0.01	0.02	0.03	0.04	0.05	0.07	0.08	0.09	0.1	0.3	0.5	0.7	0.9	1.5	5
DB-OLS	0.014356	0.014356	0.014356	0.014356	0.014356	0.014356	0.014356	0.014356	0.014356	0.014356	0.014356	0.014356	0.014356	0.014356	0.014356	0.014356	0.014356	0.014356	0.014356	0.014356	0.014356
DB-ORR	0.014356	0.014348	0.014325	0.01432	0.014316	0.014309	0.014307	0.014308	0.01436	0.014462	0.014613	0.015066	0.015366	0.015715	0.016112	0.033999	0.069725	0.121814	0.188911	0.468291	3.415822
DB-URR	0.014356	0.014349	0.014325	0.01432	0.014316	0.014309	0.014307	0.014307	0.014357	0.014455	0.014602	0.015041	0.015333	0.015673	0.01606	0.033498	0.068351	0.119185	0.184679	0.457476	3.339017
DB-MUR	0.014356	0.014342	0.014307	0.014303	0.014301	0.014303	0.014308	0.014459	0.014807	0.015352	0.016093	0.018154	0.019472	0.020981	0.022679	0.094487	0.232308	0.427993	0.674303	1.656164	10.21563
Best		DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-ORR	DB-URR	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS
ρ=0.9																					
k	0	0.001	0.005	0.006	0.007	0.009	0.01	0.02	0.03	0.04	0.05	0.07	0.08	0.09	0.1	0.3	0.5	0.7	0.9	1.5	5
DB-OLS	0.027361	0.027361	0.027361	0.027361	0.027361	0.027361	0.027361	0.027361	0.027361	0.027361	0.027361	0.027361	0.027361	0.027361	0.027361	0.027361	0.027361	0.027361	0.027361	0.027361	0.027361
DB-ORR	0.027361	0.027338	0.027262	0.027248	0.027235	0.027215	0.027208	0.027233	0.027434	0.027811	0.028361	0.029979	0.031043	0.032275	0.033675	0.094725	0.212209	0.377633	0.583738	1.388241	7.936972
DB-URR	0.027361	0.027338	0.027263	0.027249	0.027236	0.027216	0.027209	0.027232	0.027429	0.027799	0.028341	0.029935	0.030983	0.032198	0.033578	0.093798	0.209727	0.372998	0.576458	1.370896	7.845942
DB-MUR	0.027361	0.027317	0.027208	0.027199	0.027197	0.027213	0.027232	0.027807	0.029078	0.031036	0.033674	0.040952	0.045578	0.05085	0.05676	0.297723	0.736489	1.328494	2.038026	4.612076	20.24754
Best		DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-ORR	DB-URR	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS
ρ=0.95																					
k	0	0.001	0.005	0.006	0.007	0.009	0.01	0.02	0.03	0.04	0.05	0.07	0.08	0.09	0.1	0.3	0.5	0.7	0.9	1.5	5
DB-OLS	0.053579	0.053579	0.053579	0.053579	0.053579	0.053579	0.053579	0.053579	0.053579	0.053579	0.053579	0.053579	0.053579	0.053579	0.053579	0.053579	0.053579	0.053579	0.053579	0.053579	0.053579
DB-ORR	0.053579	0.053495	0.053228	0.053178	0.053135	0.053069	0.053047	0.053185	0.053984	0.055432	0.05752	0.063575	0.067523	0.072071	0.077211	0.290879	0.675511	1.184103	1.781672	3.876514	15.76154
DB-URR	0.053579	0.053496	0.05323	0.053181	0.053138	0.053072	0.053049	0.053185	0.053976	0.055411	0.057481	0.063485	0.0674	0.071911	0.077008	0.288985	0.670667	1.17546	1.768675	3.848985	15.66637
DB-MUR	0.053579	0.053419	0.053048	0.053022	0.053023	0.053105	0.053185	0.055437	0.060272	0.067629	0.077449	0.104248	0.121115	0.14022	0.16151	0.973807	2.315432	3.965787	5.780613	11.42166	32.54772
Best		DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-ORR	DB-ORR	DB-ORR	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS
ρ=0.99																					
k	0	0.001	0.005	0.006	0.007	0.009	0.01	0.02	0.03	0.04	0.05	0.07	0.08	0.09	0.1	0.3	0.5	0.7	0.9	1.5	5
DB-OLS	0.258777	0.258777	0.258777	0.258777	0.258777	0.258777	0.258777	0.258777	0.258777	0.258777	0.258777	0.258777	0.258777	0.258777	0.258777	0.258777	0.258777	0.258777	0.258777	0.258777	0.258777
DB-ORR	0.258777	0.257356	0.253338	0.252744	0.252311	0.251925	0.251969	0.260827	0.284188	0.320974	0.370189	0.502279	0.583503	0.673842	0.772612	3.790967	7.428705	10.89439	13.99998	21.22391	37.17328
DB-URR	0.258777	0.257359	0.253348	0.252754	0.252321	0.251934	0.251977	0.260803	0.284095	0.320778	0.369859	0.501598	0.582609	0.672714	0.77123	3.78235	7.41235	10.87152	13.97193	21.18615	37.12421
DB-MUR	0.258777	0.256103	0.251953	0.252514	0.253699	0.257916	0.260933	0.322777	0.437708	0.599802	0.803737	1.318478	1.621111	1.949142	2.299431	11.10396	19.25575	25.52315	30.21621	38.57101	48.26364
Best		DB-MUR	DB-MUR	DB-MUR	DB-ORR	DB-ORR	DB-ORR	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS

n=50 σ=1																					
ρ=0.8																					
k	0	0.001	0.005	0.006	0.007	0.009	0.01	0.02	0.03	0.04	0.05	0.07	0.08	0.09	0.1	0.3	0.5	0.7	0.9	1.5	5
DB-OLS	0.387257	0.387257	0.387257	0.387257	0.387257	0.387257	0.387257	0.387257	0.387257	0.387257	0.387257	0.387257	0.387257	0.387257	0.387257	0.387257	0.387257	0.387257	0.387257	0.387257	0.387257
DB-ORR	0.387257	0.387165	0.386801	0.386711	0.386622	0.386445	0.386358	0.385512	0.384719	0.383979	0.383292	0.382075	0.381545	0.381066	0.380639	0.382592	0.40339	0.441468	0.495393	0.739626	3.562142
DB-URR	0.387257	0.387165	0.386803	0.386713	0.386625	0.386449	0.386361	0.385518	0.384726	0.383986	0.383298	0.382075	0.38154	0.381055	0.380621	0.382191	0.40218	0.439062	0.49144	0.729243	3.48636
DB-MUR	0.387257	0.387073	0.386359	0.386186	0.386015	0.385679	0.385514	0.383979	0.382652	0.381531	0.380614	0.379389	0.379079	0.378967	0.379054	0.420458	0.531013	0.702167	0.926341	1.853467	10.26834
Best		DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-URR	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS
ρ=0.9																					
k	0	0.001	0.005	0.006	0.007	0.009	0.01	0.02	0.03	0.04	0.05	0.07	0.08	0.09	0.1	0.3	0.5	0.7	0.9	1.5	5
DB-OLS	0.732282	0.732282	0.732282	0.732282	0.732282	0.732282	0.732282	0.732282	0.732282	0.732282	0.732282	0.732282	0.732282	0.732282	0.732282	0.732282	0.732282	0.732282	0.732282	0.732282	0.732282
DB-ORR	0.732282	0.732021	0.730996	0.730744	0.730495	0.730002	0.729758	0.727429	0.725291	0.723344	0.721586	0.718632	0.717432	0.716415	0.715579	0.734756	0.815125	0.947422	1.123779	1.854806	8.193696
DB-URR	0.732282	0.732021	0.730996	0.730744	0.730495	0.730002	0.729758	0.727425	0.725283	0.723328	0.721561	0.71858	0.717363	0.716327	0.715471	0.733804	0.812613	0.942758	1.116475	1.837477	8.103038
DB-MUR	0.732282	0.731762	0.729757	0.729275	0.7288	0.727874	0.727422	0.723318	0.71996	0.71734	0.715448	0.713814	0.714054	0.714987	0.716605	0.880409	1.255813	1.795079	2.460183	4.935689	20.36576
Best		DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-URR	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS

Table 1. EMSE values of the estimators for when n=50.

ρ=0.95

k	0	0.001	0.005	0.006	0.007	0.009	0.01	0.02	0.03	0.04	0.05	0.07	0.08	0.09	0.1	0.3	0.5	0.7	0.9	1.5	5
DB-OLS	1.432381	1.432381	1.432381	1.432381	1.432381	1.432381	1.432381	1.432381	1.432381	1.432381	1.432381	1.432381	1.432381	1.432381	1.432381	1.432381	1.432381	1.432381	1.432381	1.432381	1.432381
DB-ORR	1.432381	1.431341	1.427262	1.426262	1.425271	1.423311	1.422343	1.413097	1.404628	1.396925	1.389976	1.378288	1.373527	1.369471	1.366111	1.428956	1.692031	2.10057	2.614357	4.523743	16.01184
DB-URR	1.432381	1.431341	1.427262	1.426262	1.42527	1.42331	1.422342	1.413089	1.404608	1.396888	1.389917	1.378171	1.373373	1.369276	1.365869	1.426956	1.687026	2.091721	2.601119	4.495902	15.91637
DB-MUR	1.432381	1.430309	1.422332	1.420415	1.418528	1.414848	1.413053	1.396767	1.383448	1.373027	1.365433	1.358455	1.35894	1.36199	1.367542	1.920433	3.078937	4.594741	6.307507	11.75499	32.61001
Best		DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-URR	DB-URR	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS

ρ=0.99

k	0	0.001	0.005	0.006	0.007	0.009	0.01	0.02	0.03	0.04	0.05	0.07	0.08	0.09	0.1	0.3	0.5	0.7	0.9	1.5	5
DB-OLS	6.914961	6.914961	6.914961	6.914961	6.914961	6.914961	6.914961	6.914961	6.914961	6.914961	6.914961	6.914961	6.914961	6.914961	6.914961	6.914961	6.914961	6.914961	6.914961	6.914961	6.914961
DB-ORR	6.914961	6.889505	6.790823	6.766924	6.743328	6.697038	6.674339	6.463099	6.27892	6.119693	5.983501	5.773371	5.696371	5.636252	5.591781	6.728522	9.428366	12.34663	15.10159	21.78988	37.2361
DB-URR	6.914961	6.889509	6.790839	6.766942	6.743348	6.69706	6.67436	6.463102	6.278868	6.119551	5.983238	5.772782	5.695582	5.63524	5.590526	6.720208	9.412408	12.3242	15.07401	21.75263	37.18764
DB-MUR	6.914961	6.864319	6.673166	6.628161	6.584242	6.499608	6.458867	6.106241	5.845074	5.664845	5.556304	5.522602	5.583842	5.689338	5.83404	12.47814	19.91305	25.87423	30.41719	38.61828	48.2634
Best		DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-URR	DB-URR	DB-URR	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS

n=50 σ=5

ρ=0.8

k	0	0.001	0.005	0.006	0.007	0.009	0.01	0.02	0.03	0.04	0.05	0.07	0.08	0.09	0.1	0.3	0.5	0.7	0.9	1.5	5
DB-OLS	9.410402	9.410402	9.410402	9.410402	9.410402	9.410402	9.410402	9.410402	9.410402	9.410402	9.410402	9.410402	9.410402	9.410402	9.410402	9.410402	9.410402	9.410402	9.410402	9.410402	9.410402
DB-ORR	9.410402	9.408576	9.401285	9.399465	9.397646	9.394012	9.392197	9.374105	9.356126	9.338261	9.320507	9.285334	9.267913	9.250603	9.233402	8.911592	8.629492	8.383602	8.170787	7.703942	7.978614
DB-URR	9.410402	9.40858	9.401301	9.399484	9.397669	9.394041	9.392229	9.374168	9.356219	9.338382	9.320655	9.285533	9.268136	9.250847	9.233667	8.912036	8.629677	8.383133	8.169305	7.697581	7.91469
DB-MUR	9.410402	9.406755	9.392207	9.388581	9.384958	9.377726	9.374115	9.338238	9.302766	9.267698	9.23303	9.164884	9.1314	9.098306	9.065598	8.488547	8.045143	7.718125	7.49262	7.304792	12.64227
Best		DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-URR

ρ=0.9

k	0	0.001	0.005	0.006	0.007	0.009	0.01	0.02	0.03	0.04	0.05	0.07	0.08	0.09	0.1	0.3	0.5	0.7	0.9	1.5	5
DB-OLS	17.42537	17.42537	17.42537	17.42537	17.42537	17.42537	17.42537	17.42537	17.42537	17.42537	17.42537	17.42537	17.42537	17.42537	17.42537	17.42537	17.42537	17.42537	17.42537	17.42537	17.42537
DB-ORR	17.42537	17.41883	17.3927	17.38619	17.37968	17.36668	17.36019	17.2956	17.23161	17.16821	17.1054	16.9815	16.92041	16.85988	16.79991	15.71055	14.80746	14.06158	13.44922	12.23408	13.11126
DB-URR	17.42537	17.41883	17.39271	17.38619	17.37968	17.36668	17.36019	17.29561	17.23162	17.16821	17.10539	16.98148	16.92038	16.85984	16.79985	15.70977	14.80526	14.05739	13.44256	12.21796	13.02527
DB-MUR	17.42537	17.41229	17.36015	17.34717	17.33421	17.30836	17.29546	17.16766	17.04194	16.91827	16.79662	16.55929	16.44355	16.32972	16.21779	14.33986	13.03863	12.18408	11.6779	11.57631	22.00394
Best		DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-URR

ρ=0.95

k	0	0.001	0.005	0.006	0.007	0.009	0.01	0.02	0.03	0.04	0.05	0.07	0.08	0.09	0.1	0.3	0.5	0.7	0.9	1.5	5
DB-OLS	34.42546	34.42546	34.42546	34.42546	34.42546	34.42546	34.42546	34.42546	34.42546	34.42546	34.42546	34.42546	34.42546	34.42546	34.42546	34.42546	34.42546	34.42546	34.42546	34.42546	34.42546
DB-ORR	34.42546	34.40105	34.30381	34.27959	34.25541	34.20716	34.18309	33.94443	33.70942	33.47799	33.25008	32.80454	32.5868	32.37234	32.16109	28.53383	25.82209	23.7831	22.24937	19.6395	21.62091
DB-URR	34.42546	34.40106	34.30382	34.27961	34.25543	34.20719	34.18312	33.94449	33.7095	33.47808	33.25018	32.80465	32.58691	32.37243	32.16117	28.5328	25.8187	23.77647	22.23892	19.61597	21.53489
DB-MUR	34.42546	34.37668	34.18285	34.13472	34.08671	33.99108	33.94346	33.47413	33.01715	32.57218	32.13891	31.3062	30.90619	30.51668	30.13742	24.34754	21.03462	19.27533	18.51502	19.32243	33.95758
Best		DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-URR

ρ=0.99

k	0	0.001	0.005	0.006	0.007	0.009	0.01	0.02	0.03	0.04	0.05	0.07	0.08	0.09	0.1	0.3	0.5	0.7	0.9	1.5	5
DB-OLS	168.5077	168.5077	168.5077	168.5077	168.5077	168.5077	168.5077	168.5077	168.5077	168.5077	168.5077	168.5077	168.5077	168.5077	168.5077	168.5077	168.5077	168.5077	168.5077	168.5077	168.5077
DB-ORR	168.5077	167.9089	165.5524	164.9727	164.3968	163.2559	162.6909	157.2327	152.103	147.2745	142.723	134.3669	130.525	126.8856	123.4341	79.83741	60.17911	49.96769	44.24773	37.77974	40.00425
DB-URR	168.5077	167.9089	165.5525	164.9728	164.3969	163.256	162.691	157.233	152.1033	147.2749	142.7234	134.3672	130.5252	126.8857	123.434	79.83153	60.16623	49.94871	44.22383	37.74646	39.96053
DB-MUR	168.5077	167.3128	162.6617	161.5301	160.4107	158.2076	157.1237	146.8928	137.6723	129.3406	121.7939	108.7095	103.0278	97.83915	93.09274	48.15094	38.11879	36.37882	36.99699	40.83944	48.45618
Best		DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-URR	DB-URR

Table 1 (continued). EMSE values of the estimators for when n=50.

n=100 σ=0.1

ρ=0.8

k	0	0.001	0.005	0.006	0.007	0.009	0.01	0.02	0.03	0.04	0.05	0.07	0.08	0.09	0.1	0.3	0.5	0.7	0.9	1.5	5
DB-OLS	0.004761	0.004761	0.004761	0.004761	0.004761	0.004761	0.004761	0.004761	0.004761	0.004761	0.004761	0.004761	0.004761	0.004761	0.004761	0.004761	0.004761	0.004761	0.004761	0.004761	0.004761
DB-ORR	0.004761	0.004753	0.00472	0.004712	0.004705	0.004689	0.004682	0.004612	0.004552	0.004502	0.004462	0.004411	0.0044	0.004399	0.004407	0.006598	0.012532	0.022069	0.035074	0.093619	0.884778
DB-URR	0.004761	0.004753	0.004719	0.004711	0.004703	0.004688	0.00468	0.004608	0.004546	0.004494	0.004451	0.004396	0.004382	0.004379	0.004385	0.006509	0.012349	0.021767	0.034627	0.092595	0.877013
DB-MUR	0.004761	0.004745	0.004681	0.004666	0.004651	0.004623	0.00461	0.004498	0.004425	0.004391	0.004396	0.004523	0.004644	0.004804	0.005002	0.016868	0.043202	0.083191	0.136063	0.364856	3.10343
Best		DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-URR	DB-URR	DB-URR	DB-URR	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS

ρ=0.9

k	0	0.001	0.005	0.006	0.007	0.009	0.01	0.02	0.03	0.04	0.05	0.07	0.08	0.09	0.1	0.3	0.5	0.7	0.9	1.5	5
DB-OLS	0.008759	0.008759	0.008759	0.008759	0.008759	0.008759	0.008759	0.008759	0.008759	0.008759	0.008759	0.008759	0.008759	0.008759	0.008759	0.008759	0.008759	0.008759	0.008759	0.008759	0.008759
DB-ORR	0.008759	0.008735	0.008646	0.008624	0.008603	0.008562	0.008541	0.00836	0.008214	0.008103	0.008027	0.007981	0.008011	0.008075	0.008173	0.017215	0.039079	0.072868	0.117748	0.311429	2.531594
DB-URR	0.008759	0.008735	0.008643	0.008621	0.0086	0.008558	0.008537	0.008351	0.0082	0.008084	0.008004	0.007947	0.00797	0.008028	0.008121	0.017006	0.038645	0.072148	0.116685	0.309042	2.516299
DB-MUR	0.008759	0.008712	0.008539	0.0085	0.008461	0.008389	0.008355	0.008093	0.007972	0.007991	0.008148	0.008878	0.009449	0.010156	0.010998	0.055167	0.147652	0.283392	0.457817	1.173748	8.00744
Best		DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-URR	DB-URR	DB-URR	DB-URR	DB-URR	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS

ρ=0.95

k	0	0.001	0.005	0.006	0.007	0.009	0.01	0.02	0.03	0.04	0.05	0.07	0.08	0.09	0.1	0.3	0.5	0.7	0.9	1.5	5
DB-OLS	0.016971	0.016971	0.016971	0.016971	0.016971	0.016971	0.016971	0.016971	0.016971	0.016971	0.016971	0.016971	0.016971	0.016971	0.016971	0.016971	0.016971	0.016971	0.016971	0.016971	0.016971
DB-ORR	0.016971	0.016908	0.016669	0.016613	0.016558	0.016452	0.016401	0.015967	0.015668	0.015502	0.015468	0.015796	0.016154	0.016642	0.017258	0.055119	0.137209	0.257698	0.411512	1.031288	6.513124
DB-URR	0.016971	0.016907	0.016665	0.016608	0.016552	0.016444	0.016392	0.015949	0.015639	0.015462	0.015418	0.015721	0.016068	0.016542	0.017145	0.054637	0.136193	0.256011	0.409036	1.025922	6.486382
DB-MUR	0.016971	0.016845	0.016397	0.016298	0.016205	0.016035	0.015958	0.015481	0.015537	0.016118	0.017221	0.020964	0.023595	0.026725	0.030348	0.198747	0.525182	0.978455	1.532873	3.60884	17.51575
Best		DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-URR	DB-URR	DB-URR	DB-URR	DB-URR	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS

ρ=0.99

k	0	0.001	0.005	0.006	0.007	0.009	0.01	0.02	0.03	0.04	0.05	0.07	0.08	0.09	0.1	0.3	0.5	0.7	0.9	1.5	5
DB-OLS	0.084147	0.084147	0.084147	0.084147	0.084147	0.084147	0.084147	0.084147	0.084147	0.084147	0.084147	0.084147	0.084147	0.084147	0.084147	0.084147	0.084147	0.084147	0.084147	0.084147	0.084147
DB-ORR	0.084147	0.083373	0.080615	0.080009	0.079436	0.078389	0.077914	0.074949	0.075142	0.078384	0.084575	0.105405	0.119856	0.136876	0.156379	0.956734	2.252579	3.779296	5.393081	10.12765	26.82645
DB-URR	0.084147	0.083369	0.080593	0.079982	0.079405	0.078348	0.077869	0.074853	0.07499	0.078172	0.084297	0.104982	0.119354	0.136291	0.155708	0.953739	2.246677	3.770356	5.381187	10.10806	26.7865
DB-MUR	0.084147	0.082629	0.077884	0.077027	0.0763	0.075233	0.074891	0.078381	0.093971	0.121046	0.159023	0.265475	0.332913	0.409173	0.493797	3.392981	7.345519	11.41057	15.2276	24.44146	43.18699
Best		DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-URR	DB-URR	DB-URR	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS

n=100 σ=1

ρ=0.8

k	0	0.001	0.005	0.006	0.007	0.009	0.01	0.02	0.03	0.04	0.05	0.07	0.08	0.09	0.1	0.3	0.5	0.7	0.9	1.5	5
DB-OLS	0.149141	0.149141	0.149141	0.149141	0.149141	0.149141	0.149141	0.149141	0.149141	0.149141	0.149141	0.149141	0.149141	0.149141	0.149141	0.149141	0.149141	0.149141	0.149141	0.149141	0.149141
DB-DB-ORR	0.149141	0.149123	0.149054	0.149037	0.14902	0.148986	0.14897	0.148808	0.148657	0.148515	0.148384	0.14815	0.148048	0.147956	0.147874	0.148265	0.152434	0.16024	0.171546	0.225182	0.992416
DB-URR	0.149141	0.149123	0.149053	0.149036	0.149019	0.148985	0.148968	0.148805	0.148653	0.148509	0.148376	0.148139	0.148036	0.147942	0.147857	0.148194	0.152281	0.159978	0.17115	0.224241	0.984886
DB-MUR	0.149141	0.149106	0.148969	0.148936	0.148903	0.148838	0.148807	0.148512	0.148257	0.148041	0.147865	0.147629	0.14757	0.14755	0.147568	0.155892	0.178798	0.215465	0.265119	0.48484	3.183987
Best		DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-URR	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS

ρ=0.9

k	0	0.001	0.005	0.006	0.007	0.009	0.01	0.02	0.03	0.04	0.05	0.07	0.08	0.09	0.1	0.3	0.5	0.7	0.9	1.5	5
DB-OLS	0.277276	0.277276	0.277276	0.277276	0.277276	0.277276	0.277276	0.277276	0.277276	0.277276	0.277276	0.277276	0.277276	0.277276	0.277276	0.277276	0.277276	0.277276	0.277276	0.277276	0.277276
DB-ORR	0.277276	0.277214	0.276971	0.276911	0.276852	0.276734	0.276675	0.276111	0.275584	0.275092	0.274636	0.273832	0.273483	0.27317	0.272892	0.274556	0.289322	0.316278	0.354577	0.529914	2.673619
DB-URR	0.277276	0.277214	0.276969	0.276909	0.276849	0.276731	0.276672	0.276105	0.275573	0.275077	0.274617	0.273805	0.273451	0.273133	0.27285	0.274377	0.288937	0.315625	0.353599	0.527657	2.658605
DB-MUR	0.277276	0.277152	0.276674	0.276558	0.276443	0.276218	0.276108	0.275083	0.274202	0.273464	0.272867	0.272094	0.271918	0.27188	0.271979	0.301799	0.380838	0.503969	0.666558	1.3511	8.079596
Best		DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-URR	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS

Table 2. EMSE values of the estimators for when n=100.

										ρ=0.95											
k	0	0.001	0.005	0.006	0.007	0.009	0.01	0.02	0.03	0.04	0.05	0.07	0.08	0.09	0.1	0.3	0.5	0.7	0.9	1.5	5
DB-OLS	0.531254	0.531254	0.531254	0.531254	0.531254	0.531254	0.531254	0.531254	0.531254	0.531254	0.531254	0.531254	0.531254	0.531254	0.531254	0.531254	0.531254	0.531254	0.531254	0.531254	0.531254
DB-ORR	0.531254	0.531076	0.530379	0.530208	0.530038	0.529704	0.529538	0.527962	0.526524	0.525223	0.52406	0.522139	0.521379	0.520753	0.520259	0.536718	0.598839	0.700661	0.836993	1.410475	6.732432
DB-URR	0.531254	0.531076	0.530375	0.530203	0.530032	0.529696	0.52953	0.527944	0.526497	0.525186	0.524012	0.522068	0.521297	0.520658	0.520151	0.536252	0.597848	0.699007	0.834558	1.405169	6.705795
DB-MUR	0.531254	0.530899	0.529533	0.529206	0.528884	0.528256	0.527951	0.525199	0.522991	0.521324	0.52019	0.519502	0.519936	0.520882	0.522335	0.649884	0.939891	1.360574	1.885744	3.890131	17.61367
Best		DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-URR	DB-URR	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS
										ρ=0.99											
k	0	0.001	0.005	0.006	0.007	0.009	0.01	0.02	0.03	0.04	0.05	0.07	0.08	0.09	0.1	0.3	0.5	0.7	0.9	1.5	5
DB-OLS	2.504972	2.504972	2.504972	2.504972	2.504972	2.504972	2.504972	2.504972	2.504972	2.504972	2.504972	2.504972	2.504972	2.504972	2.504972	2.504972	2.504972	2.504972	2.504972	2.504972	2.504972
DB-ORR	2.504972	2.501437	2.487682	2.484339	2.481034	2.474538	2.471346	2.441476	2.41524	2.39252	2.373201	2.34433	2.334568	2.327789	2.323895	2.728254	3.731344	5.034865	6.474207	10.8636	27.02039
DB-URR	2.504972	2.501434	2.487664	2.484318	2.481009	2.474505	2.471309	2.441397	2.415113	2.39234	2.372963	2.343963	2.33413	2.327274	2.3233	2.725446	3.725698	5.026226	6.46264	10.84436	26.98062
DB-MUR	2.504972	2.497935	2.47128	2.464986	2.458838	2.446979	2.441266	2.391915	2.356213	2.333492	2.323112	2.336969	2.360071	2.393244	2.435983	4.696928	8.261899	12.07854	15.72934	24.68532	43.21313
Best		DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-URR	DB-URR	DB-URR	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS

										n=100	σ=5										
										ρ=0.8											
k	0	0.001	0.005	0.006	0.007	0.009	0.01	0.02	0.03	0.04	0.05	0.07	0.08	0.09	0.1	0.3	0.5	0.7	0.9	1.5	5
DB-OLS	3.627802	3.627802	3.627802	3.627802	3.627802	3.627802	3.627802	3.627802	3.627802	3.627802	3.627802	3.627802	3.627802	3.627802	3.627802	3.627802	3.627802	3.627802	3.627802	3.627802	3.627802
DB-ORR	3.627802	3.627556	3.626573	3.626328	3.626083	3.625592	3.625348	3.622905	3.620475	3.618057	3.615651	3.610877	3.608507	3.60615	3.603805	3.559412	3.519677	3.484436	3.45353	3.385318	3.561671
DB-URR	3.627802	3.627557	3.626577	3.626332	3.626087	3.625598	3.625354	3.622918	3.620494	3.618083	3.615683	3.610921	3.608557	3.606206	3.603867	3.559576	3.519917	3.484724	3.45384	3.385551	3.557998
DB-MUR	3.627802	3.627311	3.625351	3.624862	3.624373	3.623398	3.622911	3.618067	3.61327	3.60852	3.603817	3.594551	3.589987	3.58547	3.581	3.501104	3.438672	3.392771	3.362515	3.357433	5.088932
Best		DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-URR
										ρ=0.9											
k	0	0.001	0.005	0.006	0.007	0.009	0.01	0.02	0.03	0.04	0.05	0.07	0.08	0.09	0.1	0.3	0.5	0.7	0.9	1.5	5
DB-OLS	6.533332	6.533332	6.533332	6.533332	6.533332	6.533332	6.533332	6.533332	6.533332	6.533332	6.533332	6.533332	6.533332	6.533332	6.533332	6.533332	6.533332	6.533332	6.533332	6.533332	6.533332
DB-ORR	6.533332	6.532468	6.529019	6.528158	6.527298	6.525578	6.524719	6.516158	6.507649	6.499192	6.490786	6.474127	6.465874	6.457672	6.44952	6.296844	6.163034	6.046892	5.947298	5.737718	6.26478
DB-URR	6.533332	6.532469	6.52902	6.528159	6.527299	6.52558	6.524721	6.516162	6.507655	6.499199	6.490794	6.474137	6.465885	6.457683	6.449531	6.296823	6.162914	6.046609	5.946794	5.736247	6.252311
DB-MUR	6.533332	6.531605	6.524719	6.523002	6.521287	6.517864	6.516155	6.499174	6.482388	6.465796	6.449397	6.417174	6.401348	6.385712	6.370264	6.099402	5.896326	5.754528	5.668112	5.690119	10.15442
Best		DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-URR
										ρ=0.95											
k	0	0.001	0.005	0.006	0.007	0.009	0.01	0.02	0.03	0.04	0.05	0.07	0.08	0.09	0.1	0.3	0.5	0.7	0.9	1.5	5
DB-OLS	12.94437	12.94437	12.94437	12.94437	12.94437	12.94437	12.94437	12.94437	12.94437	12.94437	12.94437	12.94437	12.94437	12.94437	12.94437	12.94437	12.94437	12.94437	12.94437	12.94437	12.94437
DB-ORR	12.94437	12.94124	12.92877	12.92565	12.92254	12.91633	12.91322	12.88232	12.85167	12.82126	12.79109	12.73148	12.70204	12.67283	12.64386	12.1118	11.66291	11.28778	10.97817	10.37438	11.68645
DB-URR	12.94437	12.94124	12.92876	12.92565	12.92254	12.91632	12.91322	12.88231	12.85164	12.82123	12.79105	12.73142	12.70196	12.67274	12.64376	12.11137	11.66199	11.28624	10.9759	10.36941	11.66161
DB-MUR	12.94437	12.93812	12.91321	12.907	12.90081	12.88844	12.88228	12.82109	12.76081	12.70143	12.64293	12.52855	12.47266	12.41762	12.36343	11.4455	10.80634	10.39857	10.18284	10.38644	19.63808
Best		DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-URR	DB-URR
										ρ=0.99											
k	0	0.001	0.005	0.006	0.007	0.009	0.01	0.02	0.03	0.04	0.05	0.07	0.08	0.09	0.1	0.3	0.5	0.7	0.9	1.5	5
DB-OLS	61.05005	61.05005	61.05005	61.05005	61.05005	61.05005	61.05005	61.05005	61.05005	61.05005	61.05005	61.05005	61.05005	61.05005	61.05005	61.05005	61.05005	61.05005	61.05005	61.05005	61.05005
DB-ORR	61.05005	60.97936	60.69821	60.62832	60.55859	60.41958	60.35032	59.66612	58.99703	58.34268	57.70266	56.46418	55.86502	55.27878	54.70515	45.45396	39.30364	35.14574	32.31302	28.24744	31.24088
DB-URR	61.05005	60.97937	60.69823	60.62834	60.55861	60.41961	60.35035	59.66617	58.99711	58.34276	57.70275	56.46426	55.86508	55.27883	54.70517	45.45263	39.30003	35.13952	32.30412	28.23128	31.20495
DB-MUR	61.05005	60.9088	60.3493	60.21079	60.07281	59.79847	59.6621	58.32728	57.04347	55.80866	54.62094	52.37945	51.32226	50.30526	49.32692	35.9975	30.06927	27.79746	27.37888	30.06581	43.59947
Best		DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-URR	DB-URR

Table 2 (continued). EMSE values of the estimators for when n=100.

n=200 σ=0.1

ρ=0.8

k	0	0.001	0.005	0.006	0.007	0.009	0.01	0.02	0.03	0.04	0.05	0.07	0.08	0.09	0.1	0.3	0.5	0.7	0.9	1.5	5
DB-OLS	0.001157	0.001157	0.001157	0.001157	0.001157	0.001157	0.001157	0.001157	0.001157	0.001157	0.001157	0.001157	0.001157	0.001157	0.001157	0.001157	0.001157	0.001157	0.001157	0.001157	0.001157
DB-ORR	0.001157	0.001155	0.001149	0.001148	0.001146	0.001143	0.001142	0.001128	0.001115	0.001104	0.001095	0.001079	0.001074	0.00107	0.001067	0.001298	0.002071	0.003378	0.005213	0.013801	0.147246
DB-URR	0.001157	0.001155	0.001149	0.001148	0.001146	0.001143	0.001142	0.001127	0.001115	0.001103	0.001093	0.001078	0.001072	0.001067	0.001064	0.001294	0.00207	0.003386	0.005232	0.013883	0.148352
DB-MUR	0.001157	0.001154	0.001142	0.001139	0.001136	0.00113	0.001128	0.001104	0.001086	0.001073	0.001066	0.001068	0.001077	0.001092	0.001112	0.002666	0.006371	0.012182	0.020052	0.055581	0.568379
Best		DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-URR	DB-URR	DB-URR	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS

ρ=0.9

k	0	0.001	0.005	0.006	0.007	0.009	0.01	0.02	0.03	0.04	0.05	0.07	0.08	0.09	0.1	0.3	0.5	0.7	0.9	1.5	5
DB-OLS	0.002173	0.002173	0.002173	0.002173	0.002173	0.002173	0.002173	0.002173	0.002173	0.002173	0.002173	0.002173	0.002173	0.002173	0.002173	0.002173	0.002173	0.002173	0.002173	0.002173	0.002173
DB-ORR	0.002173	0.002169	0.002153	0.00215	0.002146	0.002139	0.002135	0.002103	0.002075	0.002053	0.002035	0.002015	0.002012	0.002014	0.002022	0.003191	0.006277	0.011229	0.017997	0.048705	0.485353
DB-URR	0.002173	0.002169	0.002153	0.002149	0.002145	0.002138	0.002135	0.002102	0.002073	0.00205	0.002032	0.002011	0.002008	0.00201	0.002017	0.003185	0.006281	0.011253	0.01805	0.048904	0.487666
DB-MUR	0.002173	0.002165	0.002135	0.002128	0.002121	0.002108	0.002102	0.002052	0.002021	0.00201	0.002019	0.002097	0.002165	0.002253	0.002361	0.008579	0.022322	0.043285	0.071174	0.193622	1.776713
Best		DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-URR	DB-URR	DB-URR	DB-URR	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS

ρ=0.95

k	0	0.001	0.005	0.006	0.007	0.009	0.01	0.02	0.03	0.04	0.05	0.07	0.08	0.09	0.1	0.3	0.5	0.7	0.9	1.5	5
DB-OLS	0.004197	0.004197	0.004197	0.004197	0.004197	0.004197	0.004197	0.004197	0.004197	0.004197	0.004197	0.004197	0.004197	0.004197	0.004197	0.004197	0.004197	0.004197	0.004197	0.004197	0.004197
DB-ORR	0.004197	0.004185	0.004141	0.00413	0.00412	0.004099	0.004089	0.004	0.003931	0.00388	0.003848	0.00384	0.003864	0.003907	0.003969	0.009033	0.021141	0.039924	0.065038	0.175154	1.543688
DB-URR	0.004197	0.004185	0.00414	0.004129	0.004119	0.004098	0.004088	0.003998	0.003927	0.003876	0.003843	0.003833	0.003857	0.003899	0.00396	0.009029	0.021165	0.040002	0.06519	0.175646	1.54827
DB-MUR	0.004197	0.004174	0.004089	0.004069	0.004051	0.004016	0.003999	0.003878	0.003831	0.003861	0.003965	0.004398	0.004726	0.005128	0.005603	0.030119	0.081696	0.158227	0.257756	0.676814	5.176795
Best		DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-URR	DB-URR	DB-URR	DB-URR	DB-URR	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS

ρ=0.99

k	0	0.001	0.005	0.006	0.007	0.009	0.01	0.02	0.03	0.04	0.05	0.07	0.08	0.09	0.1	0.3	0.5	0.7	0.9	1.5	5
DB-OLS	0.021169	0.021169	0.021169	0.021169	0.021169	0.021169	0.021169	0.021169	0.021169	0.021169	0.021169	0.021169	0.021169	0.021169	0.021169	0.021169	0.021169	0.021169	0.021169	0.021169	0.021169
DB-ORR	0.021169	0.021034	0.020542	0.020431	0.020324	0.020124	0.020031	0.019353	0.019129	0.019353	0.020019	0.022654	0.024612	0.02699	0.029782	0.164839	0.426614	0.785854	1.219732	2.807987	13.0005
DB-URR	0.021169	0.021034	0.020539	0.020427	0.02032	0.020119	0.020026	0.019343	0.019115	0.019336	0.02	0.022632	0.024589	0.026967	0.02976	0.164942	0.427008	0.786656	1.221021	2.810976	13.01228
DB-MUR	0.021169	0.020904	0.020028	0.019855	0.019701	0.019447	0.019347	0.019348	0.021135	0.024674	0.029931	0.045464	0.055676	0.067475	0.08083	0.631419	1.594097	2.825402	4.224615	8.818918	28.8966
Best		DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-URR	DB-URR	DB-URR	DB-URR	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS

n=200 σ=1

ρ=0.8

k	0	0.001	0.005	0.006	0.007	0.009	0.01	0.02	0.03	0.04	0.05	0.07	0.08	0.09	0.1	0.3	0.5	0.7	0.9	1.5	5
DB-OLS	0.066102	0.066102	0.066102	0.066102	0.066102	0.066102	0.066102	0.066102	0.066102	0.066102	0.066102	0.066102	0.066102	0.066102	0.066102	0.066102	0.066102	0.066102	0.066102	0.066102	0.066102
DB-ORR	0.066102	0.066099	0.066088	0.066085	0.066082	0.066076	0.066073	0.066045	0.066019	0.065994	0.065971	0.065928	0.065908	0.06589	0.065874	0.06583	0.06633	0.067367	0.068933	0.076731	0.205933
DB-URR	0.066102	0.066099	0.066087	0.066084	0.066082	0.066076	0.066073	0.066045	0.066018	0.065993	0.065969	0.065926	0.065906	0.065888	0.065871	0.065826	0.066329	0.067374	0.068953	0.076813	0.207039
DB-MUR	0.066102	0.066096	0.066073	0.066067	0.066062	0.066051	0.066045	0.065994	0.065948	0.065907	0.065872	0.065819	0.065801	0.065788	0.065781	0.066787	0.069953	0.075232	0.082578	0.116576	0.621579
Best		DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-URR	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS

ρ=0.9

k	0	0.001	0.005	0.006	0.007	0.009	0.01	0.02	0.03	0.04	0.05	0.07	0.08	0.09	0.1	0.3	0.5	0.7	0.9	1.5	5
DB-OLS	0.122548	0.122548	0.122548	0.122548	0.122548	0.122548	0.122548	0.122548	0.122548	0.122548	0.122548	0.122548	0.122548	0.122548	0.122548	0.122548	0.122548	0.122548	0.122548	0.122548	0.122548
DB-ORR	0.122548	0.122538	0.122498	0.122489	0.122479	0.12246	0.122451	0.122359	0.122272	0.122191	0.122114	0.121975	0.121914	0.121857	0.121805	0.121805	0.12374	0.127557	0.133206	0.160657	0.580813
DB-URR	0.122548	0.122538	0.122498	0.122489	0.122479	0.12246	0.12245	0.122358	0.122271	0.122189	0.122112	0.121973	0.121911	0.121855	0.121803	0.121806	0.123754	0.127595	0.133277	0.160884	0.583209
DB-MUR	0.122548	0.122528	0.122451	0.122432	0.122413	0.122377	0.122359	0.12219	0.122041	0.121912	0.121804	0.121646	0.121597	0.121568	0.121559	0.125465	0.136953	0.155716	0.181457	0.297765	1.852451
Best		DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-ORR	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS

Table 3. EMSE values of the estimators for when n=200.

ρ=0.95

k	0	0.001	0.005	0.006	0.007	0.009	0.01	0.02	0.03	0.04	0.05	0.07	0.08	0.09	0.1	0.3	0.5	0.7	0.9	1.5	5
DB-OLS	0.231191	0.231191	0.231191	0.231191	0.231191	0.231191	0.231191	0.231191	0.231191	0.231191	0.231191	0.231191	0.231191	0.231191	0.231191	0.231191	0.231191	0.231191	0.231191	0.231191	0.231191
DB-ORR	0.231191	0.231151	0.230993	0.230954	0.230915	0.230838	0.2308	0.230428	0.230075	0.229742	0.229428	0.228858	0.228602	0.228365	0.228147	0.227713	0.234493	0.248114	0.268223	0.364209	1.670705
DB-URR	0.231191	0.231151	0.230992	0.230953	0.230914	0.230837	0.230798	0.230425	0.230072	0.229737	0.229422	0.228851	0.228594	0.228356	0.228137	0.227704	0.234511	0.248182	0.268363	0.364684	1.675261
DB-MUR	0.231191	0.231111	0.230799	0.230723	0.230648	0.230499	0.230426	0.229739	0.229129	0.228596	0.228139	0.227455	0.227226	0.227073	0.226995	0.240748	0.282123	0.348977	0.439323	0.833514	5.243052
Best		DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-URR	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS

ρ=0.95

k	0	0.001	0.005	0.006	0.007	0.009	0.01	0.02	0.03	0.04	0.05	0.07	0.08	0.09	0.1	0.3	0.5	0.7	0.9	1.5	5
DB-OLS	1.138745	1.138745	1.138745	1.138745	1.138745	1.138745	1.138745	1.138745	1.138745	1.138745	1.138745	1.138745	1.138745	1.138745	1.138745	1.138745	1.138745	1.138745	1.138745	1.138745	1.138745
DB-ORR	1.138745	1.138033	1.135235	1.134548	1.133867	1.132519	1.131852	1.125464	1.119573	1.114174	1.109261	1.100865	1.09737	1.094337	1.091759	1.127407	1.303105	1.587288	1.955302	3.387891	13.20818
DB-URR	1.138745	1.138032	1.135232	1.134544	1.133862	1.132513	1.131845	1.125451	1.119555	1.114151	1.109234	1.100832	1.097335	1.0943	1.091721	1.127469	1.303439	1.588013	1.956502	3.390765	13.21985
DB-MUR	1.138745	1.137325	1.131845	1.130524	1.129224	1.126683	1.125442	1.114108	1.104708	1.097203	1.091557	1.085699	1.085419	1.086858	1.089984	1.460589	2.281704	3.400327	4.708881	9.119349	28.93163
Best		DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-ORR	DB-OLS	DB-OLS	DB-OLS	DB-OLS	DB-OLS

n=200 σ=5

ρ=0.8

k	0	0.001	0.005	0.006	0.007	0.009	0.01	0.02	0.03	0.04	0.05	0.07	0.08	0.09	0.1	0.3	0.5	0.7	0.9	1.5	5
DB-OLS	1.622545	1.622545	1.622545	1.622545	1.622545	1.622545	1.622545	1.622545	1.622545	1.622545	1.622545	1.622545	1.622545	1.622545	1.622545	1.622545	1.622545	1.622545	1.622545	1.622545	1.622545
DB-ORR	1.622545	1.622499	1.622314	1.622268	1.622222	1.62213	1.622084	1.621626	1.621168	1.620713	1.620259	1.619355	1.618906	1.618458	1.618012	1.609413	1.601433	1.594061	1.587289	1.570492	1.567792
DB-URR	1.622545	1.622499	1.622315	1.622269	1.622223	1.622131	1.622085	1.621627	1.62117	1.620715	1.620262	1.61936	1.618911	1.618464	1.618019	1.609439	1.601482	1.594137	1.587398	1.570723	1.569395
DB-MUR	1.622545	1.622453	1.622085	1.621993	1.621901	1.621718	1.621626	1.620714	1.619808	1.618908	1.618014	1.616244	1.615369	1.6145	1.613637	1.597653	1.584069	1.572836	1.563903	1.550421	1.814492
Best		DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-ORR

ρ=0.9

k	0	0.001	0.005	0.006	0.007	0.009	0.01	0.02	0.03	0.04	0.05	0.07	0.08	0.09	0.1	0.3	0.5	0.7	0.9	1.5	5
DB-OLS	2.996531	2.996531	2.996531	2.996531	2.996531	2.996531	2.996531	2.996531	2.996531	2.996531	2.996531	2.996531	2.996531	2.996531	2.996531	2.996531	2.996531	2.996531	2.996531	2.996531	2.996531
DB-ORR	2.996531	2.996377	2.995765	2.995612	2.99546	2.995154	2.995002	2.993479	2.991962	2.990451	2.988947	2.985956	2.98447	2.982991	2.981517	2.953313	2.927489	2.903982	2.882732	2.831953	2.859507
DB-URR	2.996531	2.996377	2.995766	2.995613	2.99546	2.995155	2.995002	2.99348	2.991964	2.990454	2.988951	2.985962	2.984477	2.982998	2.981526	2.953348	2.92756	2.904099	2.882905	2.832347	2.862418
DB-MUR	2.996531	2.996224	2.995002	2.994697	2.994392	2.993783	2.993479	2.990451	2.987447	2.984467	2.981511	2.97567	2.972785	2.969923	2.967085	2.9152	2.872351	2.838184	2.812359	2.781719	3.666325
Best		DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-ORR

ρ=0.95

k	0	0.001	0.005	0.006	0.007	0.009	0.01	0.02	0.03	0.04	0.05	0.07	0.08	0.09	0.1	0.3	0.5	0.7	0.9	1.5	5
DB-OLS	5.862735	5.862735	5.862735	5.862735	5.862735	5.862735	5.862735	5.862735	5.862735	5.862735	5.862735	5.862735	5.862735	5.862735	5.862735	5.862735	5.862735	5.862735	5.862735	5.862735	5.862735
DB-ORR	5.862735	5.862166	5.859893	5.859325	5.858758	5.857624	5.857057	5.851408	5.845785	5.840191	5.834624	5.823572	5.818087	5.81263	5.807199	5.704197	5.611516	5.528653	5.455134	5.286241	5.414915
DB-URR	5.862735	5.862166	5.859893	5.859326	5.858758	5.857625	5.857058	5.851409	5.845788	5.840195	5.834629	5.82358	5.818096	5.81264	5.807211	5.704253	5.61164	5.528868	5.455461	5.287015	5.420288
DB-MUR	5.862735	5.861597	5.857057	5.855925	5.854793	5.852534	5.851405	5.840181	5.829061	5.818045	5.807133	5.785618	5.775015	5.764514	5.754115	5.566918	5.417327	5.302585	5.220135	5.143641	7.778237
Best		DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-ORR

ρ=0.99

k	0	0.001	0.005	0.006	0.007	0.009	0.01	0.02	0.03	0.04	0.05	0.07	0.08	0.09	0.1	0.3	0.5	0.7	0.9	1.5	5
DB-OLS	27.6809	27.6809	27.6809	27.6809	27.6809	27.6809	27.6809	27.6809	27.6809	27.6809	27.6809	27.6809	27.6809	27.6809	27.6809	27.6809	27.6809	27.6809	27.6809	27.6809	27.6809
DB-ORR	27.6809	27.66707	27.61191	27.59815	27.58441	27.55698	27.54329	27.40717	27.27255	27.1394	27.0077	26.74861	26.62118	26.49515	26.37049	24.14435	22.3582	20.92866	19.79017	17.64948	18.7841
DB-URR	27.6809	27.66707	27.6119	27.59815	27.58441	27.55697	27.54328	27.40715	27.27252	27.13936	27.00766	26.74855	26.62112	26.49508	26.37042	24.14434	22.35842	20.92924	19.79119	17.65214	18.79567
DB-MUR	27.6809	27.65325	27.54319	27.51581	27.48848	27.43398	27.40681	27.13801	26.87441	26.61592	26.36246	25.87026	25.63137	25.39717	25.16758	21.4357	19.01727	17.54629	16.76193	16.88854	30.21626
Best		DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-MUR	DB-ORR

Table 3 (continued). EMSE values of the estimators for when n=200.

The simulation results presented in the tables above demonstrate that higher correlation levels lead to a significant deterioration in the performance of the DB-OLS estimator. This is evidenced by the increased Estimated Mean Squared Error (EMSE) values resulting from the exacerbated multicollinearity problem.In Table 1 $n = 50$, with $\sigma = 0.1$and across all correlation levels, the DB-MUR estimator performs best at small values of k. However, for $\sigma = 1$ and $\sigma = 5$, the DB-MUR estimator is superior for nearly all values of k.Regarding Table 2 $n = 100$, with $\sigma = 0.1$ and for all correlation levels, the DB-MUR estimator is optimal at small values of k, while the DB-URR estimator performs better at intermediate values of k. For $\sigma = 1$ and $\sigma = 5$, the DB-MUR estimator consistently outperforms the others across almost all values of k.Similarly, in Table 3 $n = 200$,with $\sigma = 0.1$ and across all correlation levels, the DB-MUR estimator is preferred at small values of k, and the DB-URR estimator is superior at intermediate values of k. When $\sigma = 1$ and $\sigma = 5$, the DB-MUR estimator remains the best choice for nearly all values of k.

To provide further theoretical interpretations of these results, the consistent superiority of the $\widehat{\boldsymbol{\beta}}_{DB-MUR}$ estimator under severe multicollinearity and high variance levels (i.e., $\sigma = 1 \; and \; \sigma = 5$) can be explained by its modified shrinkage mechanism. In contrast to the standard $\widehat{\boldsymbol{\beta}}_{DB-ORR}$, the DB-MUR estimator successfully balances the bias-variance trade-off; it mitigates the inflated variance caused by strongly correlated variables without incurring an excessive bias penalty. Meanwhile, the robust performance of the $\widehat{\boldsymbol{\beta}}_{DB-URR}$estimator at intermediate values of k perfectly reflects the theoretical bounds established in our theorems, confirming that the reduction in variance effectively outweighs the introduced bias.

5.2 A numerical example

This section presents a numerical example to illustrate the application of the theoretical concepts outlined in Section 3. Yatchew (2000) [2] generated the data, which was subsequently considered by Tabakan and Akdeniz (2010) [5] . The data originated from a 1993 survey of 81 municipal power distribution sites in Ontario, Canada. It is well-established that the partial linear model is a straightforward quasi-parametric extension of the Cobb-Douglas model. We investigate a simpler version of the Cobb-Douglas model for electricity distribution expenses

$$\text{tc} = f(cust) + \beta_1\, wage + \beta_2\, pcap + \beta_3\, puc + \beta_4\, kWh + \beta_5\, life + \beta_6\, lf + \beta_7\, kmwire + \epsilon \tag{34}$$

where tc is the logarithm of total cost per customer, cust is the logarithm of the number of customers, wage is the logarithm of the wage rate, pcap is the logarithm of the capital price, puc is a dummy variable for utility committees that provide additional services and may benefit from economies of scale, kWh is the logarithm of kilowatt-hours per customer, life is the logarithm of the remaining life of the distribution assets, lf is the logarithm of the load factor, and kilometers represents the kilometers of distribution wire [5]. It is clear that the model (34) contains both parametric and nonparametric effects. In this part, we employ the method given in this study to estimate $\boldsymbol{\beta}$, the differentiating procedure. In this paper, we look at the differencing coefficients defined in Section 5.1 . We now denote the $(81 - 3) \times 81$ differentiating matrix $\mathbf{D}$ as follows

$$\mathbf{D} = \begin{pmatrix} d_0 & d_1 & \dots & d_m & 0 & 0 & \dots & 0 \\ 0 & d_0 & d_1 & \dots & d_m & 0 & \dots & 0 \\ \dots & \dots & & & & & & \\ \dots & \dots & & & & & & \\ 0 & 0 & \dots & d_1 & \dots & d_m & 0 & 0 \\ 0 & 0 & \dots & d_0 & d_1 & \dots & d_m & 0 \\ 0 & 0 & \dots & 0 & d_0 & d_1 & \dots & d_m \end{pmatrix}$$

The matrix $\widetilde{\mathbf{X}}'\widetilde{\mathbf{X}}$contains eigenvalues $\lambda_1 = 0.1133\,, \lambda_2 = 0.2341\,, \lambda_3 = 1.5387, \lambda_4 = 2.6188, \lambda_5 = 3.4198, \lambda_6 = 15.1399, and\; \lambda_7 = 19.4871$, and a condition number $CN = \lambda_7/\lambda_1 = 171.9952$, indicating severe multicollinearity in the dataset. We want to compare the traces of the estimated

Scalar mean squared error (SMSE) matrices $\widehat{\boldsymbol{\beta}}_{DB-OLS}, \widehat{\boldsymbol{\beta}}_{DB-ORR}, \widehat{\boldsymbol{\beta}}_{DB-URR}, \widehat{\boldsymbol{\beta}}_{DB-MUR}$. Substituting β and σ^2 with their corresponding difference estimates, $\hat{\beta}_{DB-OLS}$ and $\hat{\sigma}^2$, and selecting the Ridge parameter k for a specified interval yields the estimated SMSE. The Trace of the estimated SMSE of the $\hat{\beta}_{DB-OLS}$ estimator is represented as

$$\widehat{SMSE}\left(\widehat{\boldsymbol{\beta}}_{DB-OLS}\right) = tr[\hat{\sigma}^2 UCU] \tag{35}$$

where $\mathbf{U} = \left(\widetilde{\mathbf{X}}'\widetilde{\mathbf{X}}\right)^{-\mathbf{1}}$

The estimated SMSE of the $\hat{\beta}_{DB-ORR}$ estimator is represented as

$$\widehat{SMSE}\left(\hat{\beta}_{DB-ORR}\right) = tr[\mathbf{C_k}[\hat{\sigma}^2\mathbf{C} + k^2\widehat{\boldsymbol{\beta}}_{DB-OLS}\widehat{\boldsymbol{\beta}}_{DB-OLS}']\mathbf{C_k}] \tag{36}$$

The estimated SMSE of the $\hat{\beta}_{DB-URR}$ estimator is represented as

$$\widehat{SMSE}\left(\hat{\beta}_{DB-URR}\right) = tr[\hat{\sigma}^2\mathbf{C_k}[\mathbf{C} + k\mathbf{I}]\mathbf{C_k}] \tag{37}$$

The estimated SMSE of the estimator, $\hat{\beta}_{DB-MUR}$ is expressed as follows

$$\widehat{SMSE}\left(\hat{\beta}_{DB-MUR}\right) = tr[\hat{\sigma}^2\mathbf{W}\mathbf{C_k}[\mathbf{C} + k\mathbf{I}]\mathbf{C_k}\mathbf{W}' + k^2\mathbf{C_k}\widehat{\boldsymbol{\beta}}_{DB-OLS}\widehat{\boldsymbol{\beta}}_{DB-OLS}'\mathbf{C_k}] \tag{38}$$

Table 4. Estimated variance and SMSE values for the $\hat{\beta}_{DB-OLS}(k = 0)$ and $\hat{\beta}_{DB-ORR}$ estimators

Coef	k=0	k=0.001	k=0.005	k=0.006	k=0.007	k=0.009	k=0.01	k=0.02	k=0.03	k=0.04	k=0.05	k0.07	k=0.08	k=0.09	k=0.1	k=0.3	k=0.5	k=0.7	k=0.9	k=1.5	k=5
Wage	0.622935	0.618396	0.600923	0.596718	0.592575	0.584469	0.580504	0.543774	0.511622	0.483209	0.457895	0.414671	0.396049	0.379058	0.363486	0.199411	0.1365	0.103161	0.082564	0.051002	0.015063
Pcap	0.545191	0.544512	0.541876	0.541237	0.540604	0.53936	0.538748	0.532967	0.527717	0.522899	0.51844	0.510381	0.506699	0.503209	0.499886	0.451939	0.419408	0.393841	0.372688	0.325402	0.209156
Puc	-0.07451	-0.07463	-0.0751	-0.07522	-0.07533	-0.07555	-0.07566	-0.07666	-0.07754	-0.07833	-0.07904	-0.08028	-0.08082	-0.08131	-0.08177	-0.087	-0.08932	-0.09064	-0.09144	-0.09231	-0.08628
kWh	0.00819	0.008988	0.01205	0.012784	0.013506	0.014914	0.015602	0.021917	0.027364	0.032112	0.036288	0.043296	0.046267	0.04895	0.051387	0.07553	0.083583	0.087038	0.088467	0.087867	0.067549
Life	-0.6277	-0.62722	-0.62532	-0.62485	-0.62438	-0.62344	-0.62298	-0.61839	-0.61391	-0.60952	-0.60522	-0.59686	-0.59278	-0.58877	-0.58482	-0.51599	-0.46146	-0.41703	-0.38015	-0.29953	-0.13019
Lf	1.32687	1.314893	1.269036	1.258059	1.247266	1.226218	1.215953	1.121885	1.041118	0.971032	0.909651	0.807265	0.764135	0.725314	0.690191	0.348485	0.232	0.173546	0.138485	0.086035	0.026407
Kmwire	0.412594	0.412321	0.411232	0.41096	0.410688	0.410146	0.409874	0.407176	0.404503	0.401857	0.399239	0.394094	0.391566	0.389068	0.3866	0.342788	0.307326	0.27791	0.253067	0.197211	0.071891
$\widehat{Var}$	0.367754	0.362914	0.344644	0.340331	0.336114	0.327956	0.324008	0.288793	0.259934	0.235924	0.215685	0.183573	0.170667	0.159368	0.149407	0.065939	0.043211	0.032731	0.026599	0.017266	0.005364
$\widehat{mse}$	0.367754	0.36308	0.348507	0.345802	0.343437	0.33967	0.338238	0.337536	0.354917	0.383597	0.419203	0.500706	0.543446	0.586252	0.628565	1.233287	1.539012	1.724955	1.854077	2.093096	2.546244
R^2	0.652837	0.652833	0.652736	0.652694	0.652645	0.65253	0.652464	0.651553	0.650322	0.648907	0.647393	0.644264	0.642704	0.641167	0.639663	0.616386	0.599732	0.585423	0.572413	0.538848	0.431262

Table 5. Estimated variance and SMSE values for the $\hat{\beta}_{DB-OLS}(k = 0)$ and $\hat{\beta}_{DB-URR}$ estimators

Coef	k=0	k=0.001	k=0.005	k=0.006	k=0.007	k=0.009	k=0.01	k=0.02	k=0.03	k=0.04	k=0.05	k0.07	k=0.08	k=0.09	k=0.1	k=0.3	k=0.5	k=0.7	k=0.9	k=1.5	k=5
Wage	0.622935	0.622935	0.622935	0.622935	0.622935	0.622935	0.622935	0.622935	0.622935	0.622935	0.622935	0.622935	0.622935	0.622935	0.622935	0.622935	0.622935	0.622935	0.622935	0.622935	0.622935
Pcap	0.545191	0.545191	0.545191	0.545191	0.545191	0.545191	0.545191	0.545191	0.545191	0.545191	0.545191	0.545191	0.545191	0.545191	0.545191	0.545191	0.545191	0.545191	0.545191	0.545191	0.545191
Puc	-0.07451	-0.07451	-0.07451	-0.07451	-0.07451	-0.07451	-0.07451	-0.07451	-0.07451	-0.07451	-0.07451	-0.07451	-0.07451	-0.07451	-0.07451	-0.07451	-0.07451	-0.07451	-0.07451	-0.07451	-0.07451
kWh	0.00819	0.00819	0.00819	0.00819	0.00819	0.00819	0.00819	0.00819	0.00819	0.00819	0.00819	0.00819	0.00819	0.00819	0.00819	0.00819	0.00819	0.00819	0.00819	0.00819	0.00819
Life	-0.6277	-0.6277	-0.6277	-0.6277	-0.6277	-0.6277	-0.6277	-0.6277	-0.6277	-0.6277	-0.6277	-0.6277	-0.6277	-0.6277	-0.6277	-0.6277	-0.6277	-0.6277	-0.6277	-0.6277	-0.6277
Lf	1.32687	1.32687	1.32687	1.32687	1.32687	1.32687	1.32687	1.32687	1.32687	1.32687	1.32687	1.32687	1.32687	1.32687	1.32687	1.32687	1.32687	1.32687	1.32687	1.32687	1.32687
Kmwire	0.412594	0.412594	0.412594	0.412594	0.412594	0.412594	0.412594	0.412594	0.412594	0.412594	0.412594	0.412594	0.412594	0.412594	0.412594	0.412594	0.412594	0.412594	0.412594	0.412594	0.412594
$\widehat{Var}$	0.367754	0.365325	0.355976	0.353726	0.351511	0.347176	0.345056	0.325424	0.308251	0.293082	0.279571	0.256502	0.246561	0.237487	0.229167	0.140697	0.105551	0.086089	0.073499	0.052597	0.022261
$\widehat{mse}$	0.367754	0.365325	0.355976	0.353726	0.351511	0.347176	0.345056	0.325424	0.308251	0.293082	0.279571	0.256502	0.246561	0.237487	0.229167	0.140697	0.105551	0.086089	0.073499	0.052597	0.022261
R^2	0.652837	0.652837	0.652837	0.652837	0.652837	0.652837	0.652837	0.652837	0.652837	0.652837	0.652837	0.652837	0.652837	0.652837	0.652837	0.652837	0.652837	0.652837	0.652837	0.652837	0.652837

Table 6. Estimated variance and SMSE values for the $\hat{\beta}_{DB-OLS}(k = 0)$ and $\hat{\beta}_{DB-MUR}$ estimators

Coef	k=0	k=0.001	k=0.005	k=0.006	k=0.007	k=0.009	k=0.01	k=0.02	k=0.03	k=0.04	k=0.05	k0.07	k=0.08	k=0.09	k=0.1	k=0.3	k=0.5	k=0.7	k=0.9	k=1.5	k=5
Wage	0.622935	0.618396	0.600923	0.596718	0.592575	0.584469	0.580504	0.543774	0.511622	0.483209	0.457895	0.414671	0.396049	0.379058	0.363486	0.199411	0.1365	0.103161	0.082564	0.051002	0.015063
Pcap	0.545191	0.544512	0.541876	0.541237	0.540604	0.53936	0.538748	0.532967	0.527717	0.522899	0.51844	0.510381	0.506699	0.503209	0.499886	0.451939	0.419408	0.393841	0.372688	0.325402	0.209156
Puc	-0.07451	-0.07463	-0.0751	-0.07522	-0.07533	-0.07555	-0.07566	-0.07666	-0.07754	-0.07833	-0.07904	-0.08028	-0.08082	-0.08131	-0.08177	-0.087	-0.08932	-0.09064	-0.09144	-0.09231	-0.08628
kWh	0.00819	0.008988	0.01205	0.012784	0.013506	0.014914	0.015602	0.021917	0.027364	0.032112	0.036288	0.043296	0.046267	0.04895	0.051387	0.07553	0.083583	0.087038	0.088467	0.087867	0.067549
Life	-0.6277	-0.62722	-0.62532	-0.62485	-0.62438	-0.62344	-0.62298	-0.61839	-0.61391	-0.60952	-0.60522	-0.59686	-0.59278	-0.58877	-0.58482	-0.51599	-0.46146	-0.41703	-0.38015	-0.29953	-0.13019
Lf	1.32687	1.314893	1.269036	1.258059	1.247266	1.226218	1.215953	1.121885	1.041118	0.971032	0.909651	0.807265	0.764135	0.725314	0.690191	0.348485	0.232	0.173546	0.138485	0.086035	0.026407
Kmwire	0.412594	0.412321	0.411232	0.41096	0.410688	0.410146	0.409874	0.407176	0.404503	0.401857	0.399239	0.394094	0.391566	0.389068	0.3866	0.342788	0.307326	0.27791	0.253067	0.197211	0.071891
$\widehat{Var}$	0.367754	0.360523	0.33374	0.327538	0.321519	0.309998	0.304485	0.257053	0.220597	0.191977	0.169097	0.135214	0.122459	0.111713	0.102569	0.03887	0.025328	0.019244	0.015581	0.009742	0.002478
$\widehat{mse}$	0.367754	0.360688	0.337603	0.333009	0.328842	0.321713	0.318715	0.305796	0.315579	0.33965	0.372616	0.452347	0.495239	0.538596	0.581727	1.206218	1.521129	1.711469	1.843059	2.085572	2.543359
R^2	0.652837	0.652833	0.652736	0.652694	0.652645	0.65253	0.652464	0.651553	0.650322	0.648907	0.647393	0.644264	0.642704	0.641167	0.639663	0.616386	0.599732	0.585423	0.572413	0.538848	0.431262

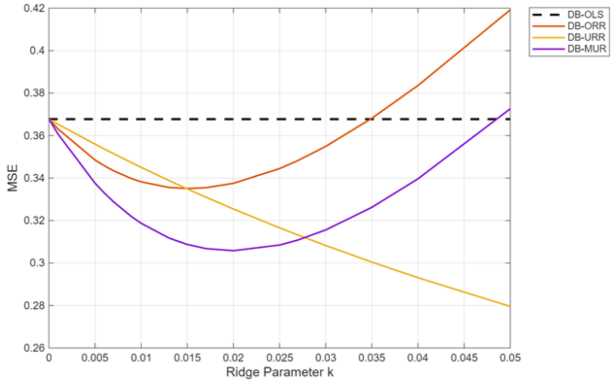

Fig. 1. Estimated SMSE values of the estimators $\hat{\beta}_{DB-OLS}(k=0)$, $\hat{\beta}_{DB-ORR}$, and $\hat{\beta}_{DB-URR}$, and $\hat{\beta}_{DB-MUR}$ for varied $k>0$

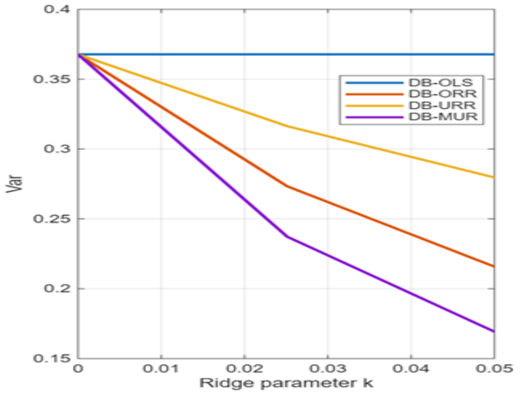

Fig. 2. Estimated variance value of the estimators $\widehat{\boldsymbol{\beta}}_{DB-OLS}(k=0)$, $\widehat{\boldsymbol{\beta}}_{DB-ORR}$, and $\widehat{\boldsymbol{\beta}}_{DB-URR}$, and $\widehat{\boldsymbol{\beta}}_{DB-MUR}$ for varied $k>0$

Fig. 1. displays the estimated SMSE values for the estimators $\widehat{\boldsymbol{\beta}}_{DB-OLS}$, $\widehat{\boldsymbol{\beta}}_{DB-ORR}$, $\widehat{\boldsymbol{\beta}}_{DB-URR}$,and ,$\widehat{\boldsymbol{\beta}}_{DB-MUR}$. The $\widehat{\boldsymbol{\beta}}_{DB-MUR}$ estimator outperforms the other estimators in the range $0<k\leq 0{,}02$, whereas the $\hat{\beta}_{DB-URR}$ estimator demonstrates superior performance for $k\geq 0.03$. These findings are consistent with Theorems 4.1, 4.2, and 4.3. Furthermore, Fig. 2. illustrates the estimated variance values for the estimators $\widehat{\boldsymbol{\beta}}_{DB-OLS}$, $\widehat{\boldsymbol{\beta}}_{DB-ORR}$, $\widehat{\boldsymbol{\beta}}_{DB-URR}$ *and* $\hat{\beta}_{DB-MUR}$.

6. Conclusions

In this paper, we propose two new estimators called the Difference-Based Unbiased Ridge Estimator ($\widehat{\boldsymbol{\beta}}_{DB-URR}$) and the Difference-Based Modified Unbiased Ridge Estimator ($\widehat{\boldsymbol{\beta}}_{DB-MUR}$). These estimators are designed to be applied in the presence of multicollinearity within partial linear models. A comprehensive comparison was conducted in terms of the Mean Squared Error Matrix (MSEM) between the two new estimators and both the Difference-Based Ordinary Least Squares estimator ($\widehat{\boldsymbol{\beta}}_{DB-OLS}$) and the Difference-Based Ordinary Ridge Estimator ($\widehat{\boldsymbol{\beta}}_{DB-ORR}$). Furthermore, a simulation study and a numerical example are presented to demonstrate the performance of the proposed estimators based on the MSEM criterion. Theoretical results indicate that the $\widehat{\boldsymbol{\beta}}_{DB-MUR}$ estimator outperforms the other estimators when small values of the ridge parameter k are selected, whereas the $\widehat{\boldsymbol{\beta}}_{DB-URR}$ estimator proves superior for intermediate and large values of the ridge parameter k.

Acknowledgment

The authors would also like to thank the University of Anbar for its support, and the editorial board and reviewers for their valuable comments and suggestions, which greatly enhanced the quality and clarity of the overall manuscript.

References

1. A. J. E. l. Yatchew, "An elementary estimator of the partial linear model," vol. 57, no. 2, pp. 135-143, 1997.
2. A. J. J. o. a. E. Yatchew, "Scale economies in electricity distribution: A semiparametric analysis," vol. 15, no. 2, pp. 187-210, 2000.
3. L. Wang, L. Brown, and T. Cai, "A difference-based approach to semiparametric partial linear model. Department of Statistics, The Wharton School University of Pennsylvania," Pennsylvania, Technical report2007.
4. A. E. Hoerl and R. W. J. T. Kennard, "Ridge regression: Biased estimation for nonorthogonal problems," vol. 12, no. 1, pp. 55-67, 1970.
5. G. Tabakan and F. J. S. P. Akdeniz, "Difference-based ridge estimator of parameters in partial linear model," vol. 51, no. 2, pp. 357-368, 2010.
6. R. H. Crouse, C. Jin, R. J. C. i. S.-T. Hanumara, and Methods, "Unbiased ridge estimation with prior information and ridge trace," vol. 24, no. 9, pp. 2341-2354, 1995.
7. F. S. M. Batah, S. D. J. S. i. M. Gore, and i. Applications, "Ridge regression estimator: Combining unbiased and ordinary ridge regression methods of estimation," vol. 4, pp. 99-109, 2009.
8. A. Yatchew, *Semiparametric regression for the applied econometrician*. Cambridge University Press, 2003.
9. R. Eubank *et al.*, "Estimation in partially linear models," vol. 29, no. 1, pp. 27-34, 1998.
10. K. Klipple and R. J. F. i. h. o. D. P. M. M. A. o. t. o. o. h. r. Eubank, "Difference-based variance estimators for partially linear models," pp. 313-323, 2007.
11. G. Trenkler and H. J. S. p. Toutenburg, "Mean squared error matrix comparisons between biased estimators an overview of recent results," vol. 31, no. 1, pp. 165-179, 1990.
12. C. R. Rao and H. Toutenburg, "Linear models," in *Linear Models: Least Squares and Alternatives*: Springer, 1995, pp. 3-18.
13. J. Groß, *Linear regression*. Springer Science & Business Media, 2003.
14. G. C. McDonald and D. I. J. J. o. t. A. S. A. Galarneau, "A Monte Carlo evaluation of some ridge-type estimators," vol. 70, no. 350, pp. 407-416, 1975.
15. M. I. Alheety, H. Nayem, and B. G. J. S. Kibria, "An unbiased convex estimator depending on prior information for the classical linear regression model," vol. 8, no. 1, p. 16, 2025.
16. R. W. Farebrother, "Further results on the mean square error of ridge regression," Journal of the Royal Statistical Society: Series B, vol. 38, no. 3, pp. 248-250, 1976.