Geometry of Multi-Temporal Space-Time: A Unified Model of Subluminal, Luminal, and Superluminal Regimes

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Abstract. Previous proposals on higher temporal dimensions, like String theory, Kaluza-Klein, multi-time cosmologies etc. treat additional axes of time differently (rather specially) from the one that we perceive. They are either tied to perception, entropy, metaphysics, or curved at a quantum level. They are not considered as isotropic or interchangeable. In this proposition, we assume that each temporal axis is physically equivalent i.e. the geometry of time obeys the same tensor symmetries that govern space. We define three kinematic classes of an entity: subluminal, luminal, and super-luminal. We propose a unified framework where the dimensional structure of space-time depends on the three classes. A smooth scalar field ϕ selects the effective dimensional pair (D_s, D_t) that explains the number of spatial and temporal axes. A canonical projection-penalization term enforces the corresponding local signature in the tetrad fields. In the subluminal regime, observers experience the usual $(3+1)$ dimensional geometry. In the luminal case, entities travelling at c, they occupy an intrinsic $(2+2)$ dimensional regime with two spatial and two temporal coordinates. In this case, a photon is represented by a null world-sheet $\Sigma\gamma(\sigma,\tau) \subset \mathbb{R}^{2+2}$ satisfying intrinsic null constraints $\gamma_{AB}\partial_\alpha X^A \partial_\alpha X^B = 0$. We show that the wave-like behaviour of a photon arises from the projection of intrinsic $2D - time$ oscillations into a $1D - time$ observational frame. Furthermore, entities in the super-luminal regime are governed by $(1+3)$ dimensional system where space-like and time-like dimensions reverse or swap functions. This framework provides a mathematically consistent reformulation of photon kinematics and shows a future path toward a unified geometric theory of subluminal, luminal and super-luminal states and higher dimensions of time.

Keywords: Multiple time-like dimensions, signature field dynamics, 2D and 3D time

1. Introduction

The nature of time has long been one of the deepest puzzles in physics. Standard physical theories treat time as one-dimensional (a single coordinate along which events are ordered), plus three spatial dimensions through which matter and energy navigate. However, several authors have indeed explored the idea that more than one temporal dimension could exist with the motive to provide explanatory insights for quantum phenomena, unification of interactions, or the structure of space-time itself. Weyl's gauge geometry introduced a scaling field, which later inspired multi-signature [1], and since in his model the metric signature was not fixed, it remains a motivation for the current theory. [2] proposed a $5D$ unification with $g_{\mu\nu}$ as a $4D$ space-time metric and fifth dimension was spatial-like. However again, the structure could be generalized with metrics of signatures $(2,3), (3,2)$ etc. [3], proposed compactification of the fifth dimension like a circle S^1. (see [4]). But the idea that a dimension can be *hidden* inspired later theories where extra time dimensions might be small or inaccessible (see [5—6]; see also recent cosmological treatments of dynamical signature flips in time-dependent $\Lambda(t)$ models by [7]). Now as we dig deep into the literatures where the space-time framework allows for more than one time-like direction, it grows evident that almost all theories describe higher temporal dimensions as non-physical, constrained, structurally distinct, or are purely auxiliary degrees of freedom.

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[8], proposed two-time physics, and the manifold carries the signature $(2, d)$, but the second temporal dimension t' is removed from physical dynamics by imposing first-class gauge constraints $Q_i(x, p) = 0$. This ensures that only one effective Hamiltonian time survives. In F-theory, the $12 - dimensional$ regime allows signature $(2,10)$, but again the second time-like direction belongs only to the elliptic fibration and does not appear as a freely evolving orthogonal coordinate, and thus, the two axes t and t' are not interchangeable [9]. As mentioned earlier, signature changing Kaluza-Klein models allow metrics whose eigenvalues satisfy transitions such as $(1,4) \rightarrow (2,3)$, but the additional time dimension emerges only on measure-zero hyper-surfaces where $\det g = 0$ (see [10], for recent treatments of dynamical signature models). And thus, it did not define an extended temporal dimension with its own causal structures [5].

The common theme underlying all these frameworks is that additional temporal dimensions are fundamentally, geometrically, or conceptually different from the ordinary time dimension we perceive. They are either computational constructs, compactified directions, or gauge-redundant variables. But, from a relativistic viewpoint, the decomposition of space-time into 'time' and 'space' components is not absolute but depends on the choice of observer's 4-velocity and the associated splitting of the tangent space $T_p\mathcal{M}$ into time-like and space-like subspaces. $\mathcal{M}$ being a manifold of spacetime, no *a priori* distinction is necessary between the directions called 'spatial' or the ones called 'temporal'.

We thus posit that space and time dimensions are not geometrically different, and they are mutually compensating axes of a single conserved manifold. By 'equivalent', we mean geometrically equivalent at the level of the tangent bundle and local signature structure; physical accessibility and dynamical relevance remain regime-dependent]. In the case of spatial dimensions, there is nothing special about one axis. For example, even if the coordinate system is rotated, the reference to any object in the system is only shifted pertaining to the axes (e.g. length becomes width etc.).

Thus, for higher time dimensions too, there is no need to treat them differently. A 3D geometry of time would then mean that there are three orthogonal temporal directions – just like x, y, z in space. An entity in 3D time has temporal volume rather than temporal length. For example, an entity in a 3D time universe would be t_1 days long, t_2 days wide, and t_3 days tall. To such an object, 'time' is extended like our 'space', and 'space' flows uni-direction like our 'time'. Just like, any entity in our universe can move freely in 3D space, they can move freely in 3D time. The metric signature swaps from

$$(-,+,+,+) \rightarrow (+,-,-,-)$$

This can be modelled as a 'rotational' transformation in a 'dimensional phase space'. Table 1 summarizes how a 3D temporal universe is a mirror of the 3D spatial one. And Table 2 shows the duality of perspectives.

Table 1. Mirror Universe

Concept	**Our universe**	**3D-time universe**
Mass	Spatial density of energy	Temporal density of existence
Velocity	Change of space position per unit time	Change of time position per unit space
Force	Curvature in Space-time	Curvature in Time-space
Energy	Time flow potential	Space flow potential

Table 2. Duality of perspectives

Our Universe (3D space + 1D time)	**3D time universe (1D Space + 3D time)**
We live in space and move through time	They live in time, move through space
Our life is measured in time duration	Their life is measured in spatial length
'Causality' flows temporally	'Causality' flows spatially
We can't go back in time	They can't go back in space
We have multiple timelines	They have multiple space-lines

The primary objection to the models of space-times with more than one temporal dimension is the apparent inevitability of causality violation, since additional time-like directions may permit closed time-like curves, ultra-hyperbolic evolution, or loss of global hyperbolicity [11, 6]. However, these paradoxes arise specifically when multiple time axes coexist within a *single* kinematic regime accessible to the same class of observers.

In our current framework, we assign different temporal dimensionalities to three distinct dynamical regimes. The summary of the model with key notations are provided in Section 2. The formal model is developed in Section 3. And two major propositions are stated and proved in Section 4.

2. Notations and summary

We propose a unified framework where there are three regimes in the universal system: subluminal (where the upper limit of velocity is 'c'), where all known particles propagate; luminal (consists of all electromagnetic waves and all particles that travel only at 'c' under any measurement or reference; and superluminal (consists of all those particles whose velocity has a lower bound at 'c'). They cannot travel less than the speed of light. Each regime is enforced by the scalar dimensionality field $\phi(x)$.

Our core postulate is that **the universe always maintains four total dimensions of existence – a combination of spatial N_s and temporal N_t axes such that:**

$$\boldsymbol{N_s + N_t = 4}$$

Notations:

$\boldsymbol{\mathcal{M}}$: Underlying smooth 4-manifold (topological space-time manifold). Coordinates x^μ(indices $\mu, v = 0, \dots, 3$). Signature signs will be indicated explicitly as needed.

$\boldsymbol{g_{\mu v}(x)}$: Lorentzian metric field except where we allow signature/type changes

$\boldsymbol{c}$: speed of light (set to 1 in geometric units where convenient)

$\boldsymbol{P}$: projection operators that select active spatial/temporal subspaces.

$\boldsymbol{\phi(x)}$: scalar dimensionality field that controls local effective counts of space/time dimensions.

$\boldsymbol{D_S(x), D_t(x)} \in \{\mathbf{1, 2, 3}\}$: effective integer spatial/temporal dimensionalities at x.

World-lines: subluminal $\left(u^\mu u_\mu < 0\right)$, luminal ($null\ k^\mu k_\mu = 0$), superluminal (space-like $w^\mu u_w > 0$ when interpreted as a trajectory parameterization; see Sec. 3.3)

Planck discreetness: we model Planck-scale temporal structure as a 3-D discrete lattice of time steps (indices n_x, n_y, n_z), motivated by works proposing Planck discreetness (see Christodoulou et al., 2022).

Multi-time wave functions: when needed, we will rely on the formalism developed in multi-time wave-function literature.

Now, we put forward the detailed mathematical and conceptual framework for our model.

3. Formal Model

Let us assume that there is one underlying 4-manifold that hosts all three 'systems'. The three systems (subluminal, luminal, and superluminal) are different local regimes of the same foresaid manifold, selected by the kinematic class of a trajectory (time-like, null, space-like) and by a scalar dimensionality field $\phi(x)$ that assigns effective integer counts $D_s(x), D_t(x)$. The metric $g_{\mu v}(x)$ is the basic geometric object. The apparent change in spatial or temporal counts arises from a combination of (i) local metric signature structure, and (ii) projection (frame) fields that single out which direction/s are experienced as time versus space by a given class of trajectories. Since, our proposition relies on the fact that time is fundamentally multi-component; thus, locally there exists three independent time axes, orthogonal to one another (or, coordinate directions that can serve as time). Note that in this paper, we do not model coordinate transformation explicitly.

Macroscopically, an observer (of our conventional universe) experiences a single effective time axis, or the normal world-like projection. At Planck-scale, we define time as a discrete $3-D$ lattice (justifying choose-one unit move among three orthogonal microscopic time axes – see Sec. 3.6). We will first define projection operators that produce sub-manifolds with specified signatures. Then, we use an action principle giving covariant field equations governing $g_{\mu v}(x)$ and ϕ.

3.1. Manifold, frames and effective dimension functions

Definition 3.1.1. (Space-time manifold): Let $\mathcal{M}$ be a connected, smooth 4-dimensional differentiable manifold. We work with coordinate charts (U, x^{μ}) on $\mathcal{M}$.

Definition 3.1.2. (Metric family with local signature control): We consider a metric tensor field g that in each chart has signature $(\underbrace{-1,\dots,1}_{D_t(x)},\underbrace{+1,\dots,+1}_{D_s(x)})$ in the active sub-space, with the constraint $D_t(x) + D_s(x) = N(x) \leq 4$. Practically, we represent this as a smooth family of $rank - 4$ metric tensors that admit a decomposition via a local orthonormal frame $e^{\mu}_{A}(x)$ (frame indices $A = 0,1,2,3$) where the number of frame vectors with time-like norm equals $D_t(x)$ and the rest $D_s(x)$ are then naturally, space-like. Formally, there exists a local orthonormal co-frame $\{e^A\}$ with $g = \eta_{AB} e^A \otimes e^B$ where $\eta_{AB} = diag(\underbrace{-1,\dots,-1}_{D_t(x)},\underbrace{+1,\dots,+1}_{D_s(x)})$ on the active sub-space are suppressed by projection operators (see Def. 3.1.4)[1].

Definition 3.1.3. (Dimensionality field): We introduce a scalar field $\emptyset: \mathcal{M} \to \mathbb{R}$, and define monotone maps: $D_s(x) = \mathcal{F}_t(\emptyset(x)) \in \{1,2,3\}$ with $\mathcal{F}_s, \mathcal{F}_t$ chosen so that they satisfy our design constraint (for example $D_t + D_s = 4$ or $\alpha D_t + D_s = \beta$. We use $D_t + D_s = 4$ as the canonical choice for all x, so $(D_t, D_s) \in \{(3,1),(2,2),(1,3)\}$ only. This, in essence, neatly capture our three-system model. [The effective dimensionality functions $D_s(x)$, $D_t(x)$ are coarse-grained observables emerging from smooth suppression of frame components, rather than fundamental discontinuous fields]. A smooth realization of these regime transitions, ensuring differentiability of the action and field equations, is discussed in Appendix A.3.

Definition 3.1.4. (Projection operators): We define smooth projection operators $P_t(x) and\ P_s(x) = I - P_t(x)$ such that x project tangent vectors onto the active time and space sub-spaces respectively (see Appendix A.1 for a rigorous geometric construction). They satisfy $P_t^2 = P_t$, $P_s^2 = P_s$, $P_t, P_s = 0$; and have ranks: $rankP_t = D_t(x), rankP_s = D_s(x)$. We can realize P_t by selecting an orthonormal set of D_t time-like $1 - form\{\tau^I\}, I = 1, \dots, D_t$ and setting

$$P^{\mu}_{tv} = -\sum_{I=1}^{O_t} u^{\mu}_{I} u_{Iu}$$

With u_I orthonormal time-like frame vectors and sign conventions adjusted to yield the correction time-like projector.

Figure 1: Active Subspace Projections Across the Three Regimes

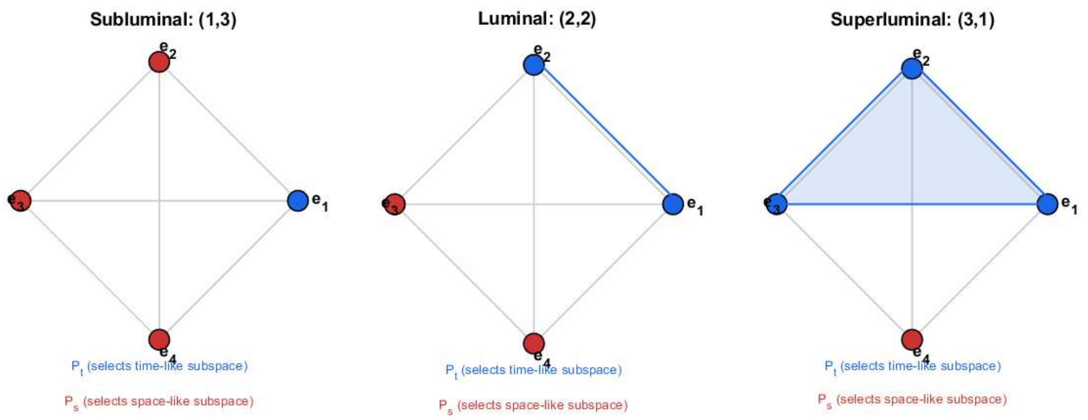


Figure 1 illustrates the active subspace selection mechanism defined in Sec. 3.1. It shows how the projection operators $P_t(x)$ and $P_s(x)$ determine the effective temporal and spatial dimensionalities

$(N_t(x), N_s(x))$ for each of the three kinematic regimes. The global manifold $\mathcal{M}$ remains 4-dimensional, but the active time-like and space-like directions with the tangent space $T_x\mathcal{M}$ depend on the regime selecting dimensionality field $\sum(x)$. Each panel shows the orthonormal field vectors $\{e_A\}_{A=1}^{4}$ at a point x. The time-like basis vectors are highlighted in blue, and the space-like in red.

3.2. Kinematic selection of the three regimes

We now formalize the idea that the type of a trajectory (world-line) determines which system it experiences. Because, based on the velocity bounds of the three regimes one entity experiences the dimensional dynamics of only one system.

Definition 3.2.1. (Kinematic class): For any smooth curve $\gamma(\lambda)$ with tangent vector $v^{\mu} = {dx^4}/{d\lambda}$ at point x, define its Lorentzian norm $N(v) = g_{\mu v} v^{\mu} v^{V}$. For subluminal system $S1$, trajectories with $N(v) < 0$ (time-like), observers following such curves experience $D_s = 3, D_t = 1$. This is the world we observe and ordinary matter evolves in. For luminal system $S2$, trajectories with $N(v) = 0$ (null), photons or all electromagnetic propagation that travels at fixed velocity 'c', experiences $D_s = 2, D_t = 2$. For superluminal system $S3$, trajectories with $N(v) > 0$ (space-like), cannot be re-parameterized as time-like, as they represent objects constrained to remain superluminal, i.e. they always experience $D_s = 1, D_t = 3$.

This follows our principle that the regimes are velocity-selected. We treat the assignment $N(v) \mapsto D_s, D_t$ as a rule enforced by the local physics implemented covariantly by the field ϕ and by coupling terms in the action (see Sec. 3.4)[2].

3.3. Metric ansatz and active subspace metric

As we keep a single global manifold, a covariant mechanism is needed to encode 'one-dimension-becoming time' and also 'two-dimensions-becoming time'. Thus, the following construct of active subspace metric is necessary:

Let $g_{\mu v}(x)$ be written via projection as

$$g_{\mu v}(x) = g_{\mu v}^{t}(x) + g_{\mu v}^{s}(x) \tag{1}$$

Where, $g_{\mu v}^{t} = -\sum_{I=1}^{D_t(x)} \tau_{\mu}^{I} \tau_{v}^{I}$, $g_{\mu v}^{s} = -\sum_{J=1}^{D_s(x)} \sigma_{\mu}^{J} \sigma_{v}^{J}$, and $\{\tau^I, \sigma^J\}$ form a local orthonormal coframe adapted to the active directions. In coordinate components, the metric remains a smooth 4×4 symmetric tensor but with the effective signature locally described by the above decomposition.

For practical implementation, we may choose a smooth background orthonormal frame e_A^{μ} and a smooth mapping $\Pi(\phi(x))$ that re-weights basis vectors so that a given subset has time-like character. For calculations, we may represent Π as a diagonal matrix with entries ± 1 or small continuous interpolations of these values. For example, if we enforce $D_s + D_t = 4$, the three allowable signatures are: $(D_t, D_s) = (1,3)$: $signature\ (- + ++)$: the usual subluminal; $(D_t, D_s) = (2,2)$: $signature(- - ++)$: luminal effective; and $(D_t, D_s) = (3,1)$: $signature(- - -+)$: superluminal effective[3].

3.4 Action principle and field equations

We now formalize a covariant action whose variation yields coupled equations for geometry and dimensionality field ϕ. The action is designed to (i) reproduce Einstein gravity in the subluminal regime (formalizing system 3 is beyond the scope of the current paper), (ii) produce explicit coupling that drives the assignment $D_s + D_t = 4$, and (iii) give dynamics to ϕ so the dimensionality configuration is a physical field s.t. evolution.

We propose the total action as:

$$S[g, \phi, \Psi] = \frac{1}{16\pi G} \int_{\mathcal{M}} d^4x \sqrt{|g|} (R[g]) + \mathcal{L}_{\phi}[g, \phi] + \mathcal{L}_{proj}[g, \phi]) + S_{matter}[g, \phi, \Psi] \tag{2}$$

Where, $R[g]$ is the Ricci scalar; $\mathcal{L}_{\phi}$ is the Lagrangian density for the scalar dimensionality field; $\mathcal{L}_{proj}$ enforces the projection structure (i.e. selects active directions and penalizes inconsistent metric/signature); S_{matter} is the matter action (also potentially includes specialized luminal or superluminal sectors); and $|g|$ denotes $|\det g_{\mu\nu}|$.

$\mathcal{L}_\phi$ is a canonical choice:

$$\mathcal{L}_\phi = -\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi - v(\phi) \qquad (3)$$

With $v(\phi)$ being a potential whose minima correspond to the three desired regimes (for example, a triple-well potential with minima $\phi = \phi_{(3,1)}, \phi_{(2,2)}, \phi_{(1,3)}$ mapping to (D_t, D_s) via $\mathcal{F}_s, \mathcal{F}_t$). The structure of the potential and the stability of distinct dimensional regimes are discussed in Appendix A.4.

Projection term $\mathcal{L}_{proj}$. We implement a covariant penalization that forces the metric to align with the desired signature count determined by ϕ. One functional is:

$$\mathcal{L}_{proj} = -\lambda \sum_{A=0}^{3} (\chi_A(\phi) g(e_A, e_A) - \eta_{AA})^2 \qquad (4)$$

Where $\{e_A\}$ is a chosen orthonormal frame field; $\eta_{AA} \in \{-1, +1\}$ are the target signs according to the desired signature at ϕ; $\chi_A(\phi)$ is a smooth selector function that tends to 1 when the A^th direction is active and to 0 when inactive; $\lambda > 0$ is a large coupling constant that enforces the constraint. This term can be understood as a penalty realization of a constrained variational problem enforcing the desired signature (see Appendix A.2).

This term makes the minimization favour frames where the metric components $g(e_A, e_A)$ take the target signs and magnitudes; in the large λ limit the metric acquires the detailed active subspace structure. Now, varying S w.r.t. $g^{\mu\nu}$ yields the modified Einstein equations:

$$G_{\mu\nu} + \Delta_{\mu\nu}[\phi, g] = 8\pi G T_{\mu\nu}^{(\Psi)} \qquad (5)$$

Where, $G_{\mu\nu} = R_{\mu\nu} - {}^1\!/_2 R g_{\mu\nu}$ and $\Delta_{\mu\nu}$ collects contributions from $\mathcal{L}_\phi$ and $\mathcal{L}_{proj}$ (explicit expressions computable from the chosen χ_A and λ)[4].

Varying w.r.t. ϕ gives:

$$\Box_g\phi - V'(\phi) + S[\phi, g] = 0 \qquad (6)$$

where S arises from coupling to the projection term (driving ϕ to the wells of V that correspond to desired dimensional regimes).

Finally, variation w.r.t. the matter field Ψ gives matter equations which may depend on ϕ (we allow the matter Lagrangian to include ϕ couplings to implement the idea that luminal excitations are constrained to $2D$ spatial behaviour etc.)[5]

3.4. World-lines, observers and measurement as projection

To model how different observers see phenomena (see Sec. 3.8 for special case detail), we formalize measurement as a projection operation. Let an observer be represented by a time-like world-line γ_0 with four-velocity u_0^μ. The observer's experience of space-time events is given by the projection onto the observer's effective $1D - time$ axis T_0, implemented by

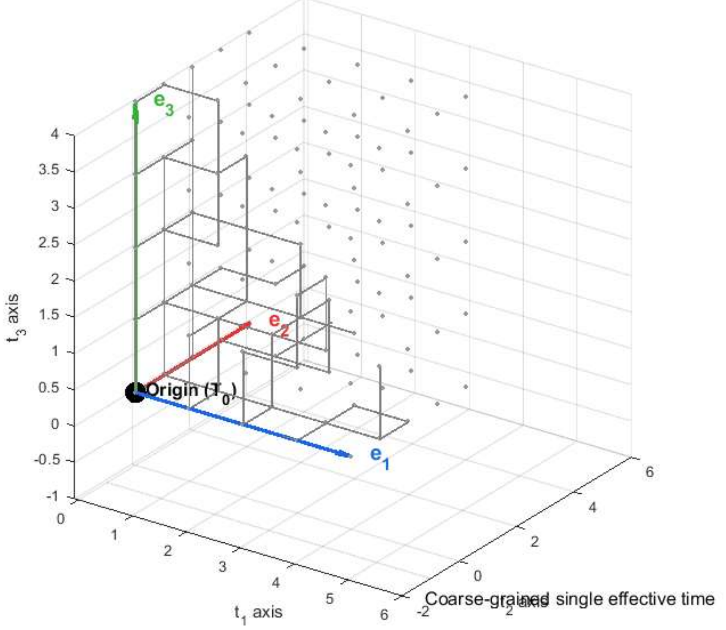
$$\Pi_0 : T_x\mathcal{M} \to span\{u_0\} \qquad (7)$$

Measurement of any field $\mathcal{F}(x)$ is modelled as the result of applying Π_0 to the field's dependence on the active time subspace. Thus, a luminal excitation which is extended in $2D - time$ subspace, when observed by a subluminal observer, will be projected onto the observer's single time axis and hence appear as an extended structure in space/time from the observer's viewpoint. [Detailed framework is provided in 3.7 and 3.8].

3.5. Planck-scale discreteness of time and the 3D time lattice

Our framework emphasizes and builds on the theory that time is discrete at Planck scale and that choices happen along discrete orthogonal steps in a $3D$ time lattice. Observe in Figure 2: Assume that entity A is currently at the origin and has three orthogonal time axes to choose from. Any conscious choice forces A to take one of the three mutually exclusive paths. Earlier studies have represented timelines as branches without any defined coordinate system (see [12]). But here, we propose that at each Planck tick, reality diverges in three orthogonal directions (t_1, t_2, t_3). To us, this seems like infinite branching (many-worlds), but it is actually finite and geometric. Analogy: Imagine a diagonal line on a pixelated screen that looks smooth but is built from discrete jumps. Furthermore, once timelines diverge, the probability of re-convergence in a three dimensional plane is negligibly small; thus, we almost never see timelines colliding.

Figure 2: Planck-Scale 3D Time Lattice and Discrete Temporal Branching



In Figure 2, the 3D time lattice is depicted as an integer grid $\mathbb{Z}^3$ where each discrete node is a Planck Time. There are three independent time axes, and each discrete step is a choice among these three. Random branches show possible microscopic timelines.

Now, we formally model this as follows:

At scales $\lesssim l_p$ (Planck length) and time steps $\lesssim t_p$ define a total discrete structure where time coordinates become integer triples $\boldsymbol{n} = (n_x, n_y, n_z) \in \mathbb{Z}^3$. A physical world-line at Planck scale is a sequence of steps:

$$\boldsymbol{n}_0 \to \boldsymbol{n}_1 \to \boldsymbol{n}_2 \dots,$$

Where each $\boldsymbol{n}_{k+1} - \boldsymbol{n}_k$ is an elementary unit vector in one of the three orthogonal time directions. Macroscopically, sequences combine to create effective continuous coarse-graining. Now, for quantum modelling (photon etc.) we use multi-time amplitudes $\Psi(\boldsymbol{n}_1, \boldsymbol{n}_2, \dots; \boldsymbol{x}_1, \boldsymbol{x}_2, \dots)$, where each particle is assigned a triple of discrete time indices and spatial coordinates. The multi-time partial differential equations follow the approach of [13] for continuous multi-time wave functions.

3.6. Quantum formalism: multi-time wave functions and measurement mappings

We define a multi-time wave function for N particles as:

$$\Psi(x_1^{(t)}, x_1^{(s)}; \dots; x_N^{(t)}, x_N^{(s)})$$

Where each $\boldsymbol{x}_1$ includes a tuple of time coordinates if $D_t > 1$.

The multi-time framework of [13] provides consistent partial differential equations:

$$i\hbar \frac{\partial}{\partial t_i} \Psi = \hat{H}_i \Psi, \quad i = 1, \dots, N. \qquad (8)$$

With consistency conditions $[\hat{H}_i, \hat{H}_j]\Psi = 0$ on solutions; for the discrete-time lattice replaced by partial differential equations. We get the required analogous consistency requirements. (See [13] for rigorous treatment and consistency constraints).

An observer associated with a single time axis selects a slice of the multi-time wave function by fixing that observer's time coordinates and marginalizing over the other time coordinates:

$$\Psi_{obs}(t_{obs}\{\boldsymbol{x}_i\}) - \sum_{oth \quad time\ indices} \Psi(\ldots)$$

Interference effects (double slit) appears when the marginalization yields coherent superpositions across different microscopic time paths. Any observation (which physically enforces a particular coarse-grained time index) reduces coherence and yields particle-like localized detection events.

3.7. Photon dynamics in effective (2+2) dimensional kinematics

We now state explicitly how a single photon can be modelled within the dimensionality framework developed. We show, how this model yields the observed wave-particle duality phenomena. Let the photon's intrinsic manifold be a $2+2$ dimensional chart with coordinates

$$X^A = (y, z; t_1, t_2), \quad A = 1, \ldots, 4$$

Where (y, z) are the two intrinsic spatial coordinates and (t_1, t_2) are the two orthogonal intrinsic temporal coordinates (all coordinates are real valued). We model the photon as a world sheet map $\sum\gamma: (\tau, \sigma) \mapsto X^A(\tau, \sigma)$ where τ, σ are the intrinsic world sheet parameters.

Standard relativity states that for a photon traveling along x:

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 = 0 \qquad (9.1)$$

Thus,

$$dt^2 = \frac{1}{c^2}(dx^2 + dy^2 + dz^2) \qquad (9.2)$$

But in the null co-moving limit of the photon (a limit construction), the longitudinal dimension satisfies: $dx' = 0$. Hence,

$$ds'^2 = dy'^2 + dz'^2 = (dt')^2 + (dt'')^2 = 0 \qquad (10)$$

And we need one more time dimension t'' to preserve nullness. This gives the photon an effective signature: $Photon: (2+2); signature = (y, z; t_1, t_2)$. We find, this already implicitly exists in string theory (world sheet formation), although interpreted differently. We find that here, it arises naturally from velocity-constrained dimensional reduction. Now, we define the photon's configuration as a world sheet:

$$\sum\gamma: (\tau, \sigma) \mapsto \big(y(\tau, \sigma), z(\tau, \sigma), t_1(\tau, \sigma), t_2(\tau, \sigma)\big) \qquad (11)$$

A photon is a $1D$ object from our projection, but a $2D$ object in its intrinsic time-space. We impose the null constraint: $\partial_\tau X^A \partial_\tau X^A = 0$; $\partial_\sigma X^A \partial_\sigma X^A = 0$. This produces the classicical $2D$ wave equation:

$$\partial_\tau^2 X^A = \partial_\sigma^2 X^A \qquad (12)$$

Thus, the photon is vibrating on a world-sheet, and its wave nature is literally its intrinsic $2D$ temporal oscillation.

Now, to define its projection to $3+1$ dimensions, we say:

$$\Pi: (y, z, t_1, t_2) \mapsto (x = t_2 c, y, z, t = t_1)$$

That is, our time is same as the photon's first time dimension, and our space-longitudinal direction is the photon's second time dimension. Thus, the wave packet we observe is the projection of motion (t_1, t_2) onto our single time.

Now, if we try to mathematically visualize a single-photon interference (it may) naturally emerge with path amplitude:

$$\Psi(x, y) = \exp\big(ikt_2(x, y)\big) \qquad (13)$$

Since oscillation in t_2 becomes spatial oscillation in our frame. If two valid photon histories correspond to different internal time paths $t_2^{(1)}, t_2^{(2)}$, then: $\Psi = \Psi_1 + \Psi_2$, and the interference term is:

$$|\Psi|^2 = 2 + 2\cos\,(k(\Delta t_2) \qquad (14)$$

Thus, we may say (subject to further scrutiny) that interference is equal to the difference in internal temporal phase. In future works, this may potentially give a geometrically emergent explanation compatible with quantum amplitudes without committing to the theory that the photon splits. We simply explore multiple Planck-time pixels in its 2-time plane. Similarly, for trillions of photons or photon beam (world sheet family), we may say that they share the same $2D$ surface $\sum\gamma^{(n)}$, and their interference is the constructive superposition of their internal t_2-oscillations. This yields (i) coherence, (ii) polarization behaviour, and (iii) classical interference fringes, because: Phase = internal 2nd time dimension. (See figure 3 for a detailed visual idea).

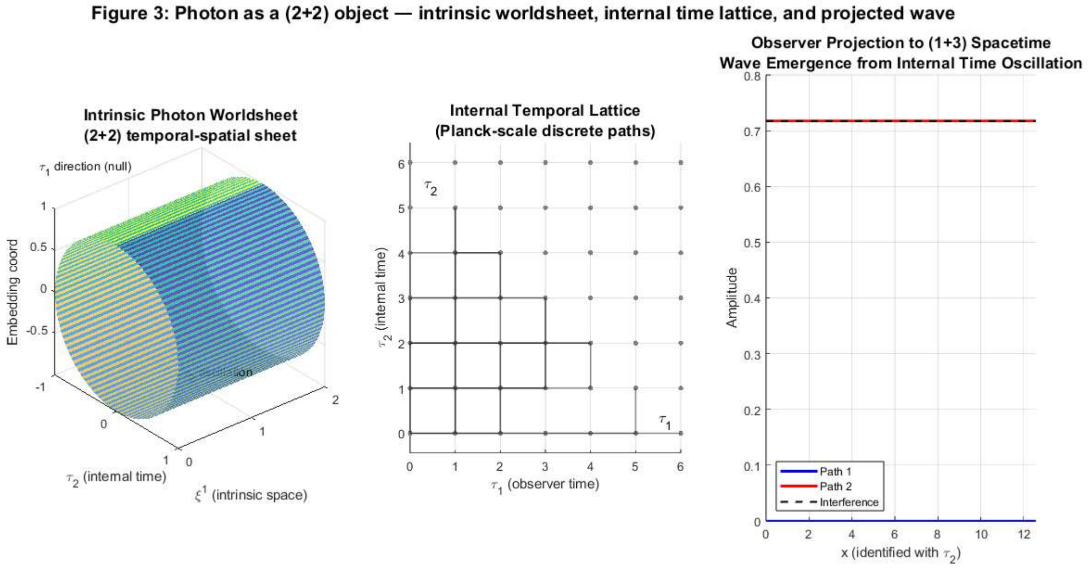


Fig. 3. Illustrates the photon as an intrinsic $(2+2)$ dimensional object. In the figure, Panel A shows its internal worksheet where oscillations occur along the second temporal axis τ_2. Panel B shows the discrete Planck scale time lattice supporting multiple microscopic temporal histories. Panel C depicts how the projection $(\tau_1, \tau_2) \mapsto (t_{obs}, x_{obs})$ produces the observed electromagnetic wave with interference arising from differences in internal temporal frame. (see [14], for recent discussions of signature transitions in cosmology).

Example. To make the construction concrete, we produce spherically symmetric model where the interior region undergoes a sequence of dimensional transitions from (3,1) to (2,2) to (1,3) as radius r decreases (motivated by asymptotic silence/dimensional reduction near singularities – (see [15]).

Ansatz. Take static, spherically symmetric coordinates (t, r, θ, φ). Define $\phi = \phi(x)$ monotone decreasing with r, with thresholds $r_1 > r_2$ such that:

- $r > r_1 : \phi \approx \phi_{(3,1)}$ (subluminal regime)
- $r_2 < r < r_1 : \phi$ crosses the (2,2) well
- $r < r_2 : \phi$ is near $\phi_{(1,3)}$

We choose metric ansatz

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + C(r)d\Omega^2 \qquad (15)$$

But allow effective reinterpretation inside r_1, r_2 where one of the angular coordinates becomes time-like (via $\mathcal{L}_{proj}$ enforcing sign change in metric components). The precise continuation requires numerical matching across thin transition shells where ϕ varies rapidly (we may use Israel junction-type matching adapted to variable signature).

4. Mathematical theorems and propositions

Proposition 4.1 (Existence of local adapted frame): Let $(\mathcal{M}, g)$ be a smooth 4-dimensional Lorentzian manifold equipped with a smooth scalar field ϕ projection penalization term $\mathcal{L}_{proj}$ is finite. Then, in any simply connected neighbourhood U where ϕ is away from threshold bifurcation points (i.e. D_t, D_s remains constant on U) there exists a smooth orthonormal frame field adapted to the (D_t, D_s) assignment.

Proof sketch (Detailed analytical proof presented in Appendix B1):

Since, g is a smooth Lorentzian metric on smooth manifold $\mathcal{M}$, at any point $p \in U$ the tangent space $T_p\mathcal{M}$ is a 4-dimensional real vector space equipped with a nondegenerate symmetric bilinear form of index D_t. And since, ϕ is smooth and remains in a single potential well on U, both $D_t(x)$ and $D_s(x)$ are constant on U. Thus the signature of g does not change within U.

Nondegeneracy and fixed index imply that every tangent space $T_x\mathcal{M}, x \in U$ admits orthonormal bases with that same signature.

At any point $x \in U$, choose any basis $\{v_A\}_{A=0}^{3}$ of the tangent space. Apply the indefinite Gram-Schmidt orthonormalization bilinear forms of fixed index.

This yields vectors $\{e_A\}$with $g(e_A, e_B) = \eta_{AB}$. [This is a classical result in pseudo-Riemannian geometry.]

Now, to extend this orthonormal basis smoothly over the entire neighbourhood U, start from any smooth frame field $\{v_A(x)\}$ on U, and apply the Gram-Schmidt algorithm point-wise. Since the Gram-Schmidt transformation involves only algebraic operations and square-roots of smooth functions of the components of $g_{\mu v}(x)$, and $g_{\mu v}(x)$ is smooth on U, the resulting frame field $\{e_A(x)\}_{A=0}^{3}$ is also smooth.

The projection penalization term

$$\mathcal{L}_{proj} = -\lambda \sum_{A} (\chi_A(\phi) g(e_A, e_A) - \eta_{AA})^2 \qquad (16)$$

Penalizes deviations of $g(e_A, e_A)$ from η_{AA}. Inside U, ϕ does not cross thresholds, so $\chi_A(\phi)$ remains constant (equal to 0 or 1 depending on whether a direction should be active or suppressed). Thus, the minimum of the penalty term corresponds exactly to the desired signature. The tetrad constructed through Gram-Schmidt with the target signature η_{AB} realizes this minimum.

[This completes the proof]

Proposition 4.2 (Well-posedness in fixed regime): In any region $U \subset \mathcal{M}$ where the effective dimensionality pair (D_t, D_s) is constant (i.e. no signature transitions occur inside U), the coupled Einstein- ϕ system derived from the action:

$$S = \int \sqrt{|g|} \left[R + \mathcal{L}_\phi + \mathcal{L}_{proj}\right] d^4x \qquad (17)$$

Forms a quasi-linear hyperbolic PDE system, once an appropriate gauge is chosen.

Proof sketch (Detailed analytical proof presented in Appendix B2):

Since, D_t, D_s remain constant on U, the metric retains a pseudo-Riemannian with fixed index $(-,+,+,+)$, $(-,-,+,+)$ or depending on the regime $(3+1), (2+2), (1+3)$. In all cases, at least one time-like direction exists. So, the system admits hyperbolic evolution.

In harmonic coordinates, the Einstein equations reduce to a second order quasi-linear hyperbolic system for $g_{\mu v}$. Concretely, in harmonic gauge

$$\Box_g \phi - V'(\phi) + S_{proj}(\phi, g) = 0 \qquad (18)$$

Since $\Box_g$ is hyperbolic for a fixed Lorentzian metric, the PDE is also hyperbolic. The projection source term S_{proj} contains algebraic functions of g, e_A, ϕ and their first derivatives; and these do not spoil hyperbolicity.

Both Einstein and scalar field equations are second order quasi-linear hyperbolic PDEs, meaning that the highest derivative terms appear linearly, but the coefficients depend smoothly on lower order derivatives or fields. The system g_μ, ϕ thus forms a closed quasi-linear hyperbolic system in a fixed signature region. Quasi-linear hyperbolic systems are known to admit local existence, uniqueness, and continuous dependence on initial data. This follows from standard theorems (e.g. local well-posedness of hyperbolic PDEs via the Leray or [16] framework).

Initial value formulation. We choose an initial space-like hyper-surface $\Sigma \subset U$, specifying (i) the induces 3-metric h_{ij} (ii) the extrinsic curvature k_{ij} (iii) the initial scalar field $\phi|_\Sigma$ and (iv) its normal derivative $n^\mu \nabla_\mu \phi|_\Sigma$. These data satisfy the usual constraint equations obtained from the (t,t) and (t,i) components of Einstein equations. Since all constraints are elliptic (as in standard GR) and the evolution system is hyperbolic, the standard GR initial value machinery applies. Since, $\mathcal{L}_{proj}$ only adds algebraic and first derivative contributions to the stress tensor, no obstruction arise from it. A standard Cauchy formulation exists.

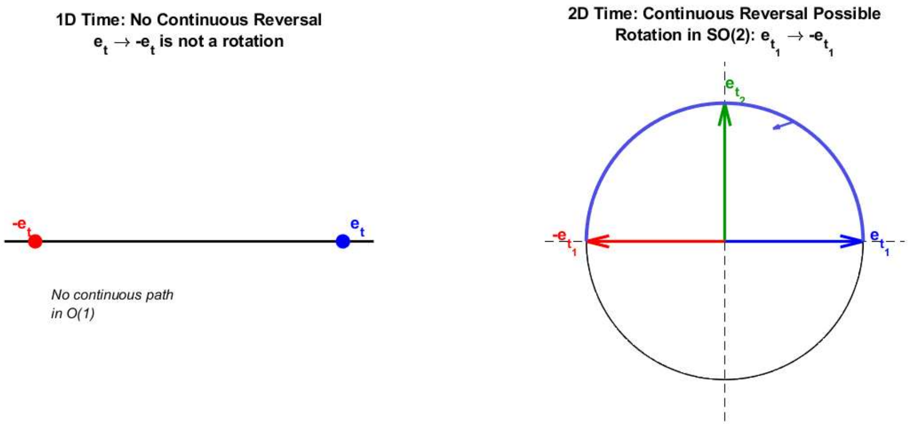
[This completes the proof.]

Claim: Under the action $S[g, \phi, \Psi]$ with projection penalty $\mathcal{L}_{proj}$ and a triple-well potential $V(\phi)$, there exist stationary, spherically symmetric solutions that exhibit a monotone transition $(3,1) \rightarrow (2,2) \rightarrow (1,3)$ as r decreases; the associated multi-time quantum amplitudes, when marginalized to an observer's single time axis yield interference patterns consistent with standard wave-particle experiments.

5. Discussion

In the conventional $(1 + 3)$ dimensional formulation of relativity, the temporal direction accessible to observers in a one-dimensional subspace spanned by a single time-like basis vector e_t. In this geometric set up, the reason why reversing temporal evolution is not possible is this: temporal reversal means a transformation $e_t \rightarrow -e_t$. But, in a one-dimensional linear subspace, the only continuous symmetry group is $O(1)$, consisting merely of the identity and a discrete sign-flip. Since, there is no continuous path in $O(1)$ that connects e_t and -e_t, temporal reversal cannot be achieved through a continuous dynamical trajectory. But if, time possesses a two dimensional (say) temporal plane $span\{e_{t_1}, e_{t_2}\}$, the symmetry group becomes $SO(2)$. This then permits continuous rotations parameterized by an angle θ. (see Fig. 4).

Fig. 4: Geometry of Temporal Reversal: Turning Around Requires a Second Time Dimension



A reversal of the effective physical time direction can be represented as a rotation:

$$e_t(\theta) = cos\theta e_{t_1} + sin\theta e_{t_1}, \qquad e_t(0) = e_{t_1}, e_t(\pi) = -e_{t_1}$$

Thus, the "turning around" of time becomes meaningful only when the temporal subspace satisfies $N_t \geq 2$. This offers a geometric reinterpretation of temporal irreversibility. The fact that subluminal observers, constrained by the dimensionality field $\phi(x)$ to an effective $(1 + 3)$ regime, lack access to the additional temporal directions required to implement such rotations.

Any multi-time frameworks (e.g. [6,8]) implicitly allow such rotational temporal degrees of freedom, but inherently suffers from the paradox of retro-causality or cause-effect violations. But the present framework explicitly shows that these degrees remain dynamically inaccessible in our (sub-luminal) regimes. Their manifestations in $N_t \geq 2$, is also bounded by the fact that the latter cannot directly interfere with our 3D spatial field except for being wave-like projections.

5.1. Potential relation to General Relativity (GR)

Although we do not posit any claim in connection to GR, we still can propose how the framework may reduce to the same – but in the limit where the dimensionality field stabilizes at $\Sigma(x) = \Sigma_{(1,3)}$, corresponding to $N_t = 1$. In this regime, the projection operators satisfy:

$$P^{\mu}_{t\ \nu} \to u^{\mu} u_{\nu}, P^{\mu}_{S\ \nu} \to \delta^{\mu}_{\nu} - u^{\mu} u_{\nu} \qquad (19)$$

Where u^{μ} is the conventional time-like four-velocity field.

Moreover, the projection Lagrangian vanishes at its minimum: $\mathcal{L}_{proj} \to 0$ and the action reduces to the Einstein-Hilbert action coupled to a scalar field with constant value.

Consequently, the Einstein equations recover their standard form:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

This ensures compatibility with experimental predictions of G.R in the $(1+3)$ regime.

5.2. Potential relation to Quantum Field Theory (QFT)

There is a deviation proposed from the way multi-time formulations are projected in relativistic quantum mechanics. In QFT, wave-functions depending on multiple time variables are introduced, subject to consistency conditions of the form $[H_i, H_j] = 0$. This ensures integrability of evolution across different temporal parameters [13]. Note that, in QFT, the additional time coordinates are formal constructs only. But we propose a geometric realization of the same. The potential bridge between our proposed geometry and quantum multi-time evolution requires a much deeper investigation.

6. Conclusion and Experimental directions

This paper's primary goal is to put forward a geometric proposal of three space-time regimes. We aim to show logical consistencies with existing theorems, but refrain from claiming experimental proofs of the framework. However, there could be two indirect empirical signatures that can be explored.

(i) Interferometric phase corrections from internal temporal structure of photons: In the luminal regime $(N_t, N_s) = (2,2)$, we observe coordinates identified as: $t_{obs} = \tau_1, x_{obs} = \tau_2$.

If the electromagnetic phase arises from an internal temporal phase $\varphi(\tau_1, \tau_2)$, then along a path γ:

$$\mathbf{\Phi}_{\gamma} = \int_{\gamma} k_{\mu} dx^{\mu} + \delta\mathbf{\Phi}[\Sigma] \qquad (20)$$

Where $\delta\mathbf{\Phi}[\Sigma]$ encodes corrections induced by the projection $P_t(\Sigma)$ and possible variations of $\Sigma(x)$ along the path.

Large scale interferometers are quite sensitive to extremely small phase-shifts. For example, kilometre-scale interferometers have been seen to resolve phase variations corresponding to strain sensitivities $\sim 10^{-21}$ [17]. In a setting with stabilized optical cavities, it could be checked if residual phase anomalies, other than environmental effects, could be parameterized as

$$\delta\mathbf{\Phi} \sim \epsilon \int \partial_{\mu} \Sigma(x) dx^{\mu} \qquad (21)$$

With ϵ being a phenomenological coulpling.

(ii) Coherence and polarization anomalies as signatures of projected temporal degrees of freedom: we have proposed that the projector $P_t(\Sigma)$ selects the active temporal subspace.

For photons, τ_2 suggests that the standard coherence functions may acquire corrections associated with internal temporal phase dispersion.

For a quasi-monochromatic field, the first order coherence function is:

$$G^{(1)}(\Delta x, \Delta t) = \langle E^{*}(x,t) E(x + \Delta x, t + \Delta t) \rangle \qquad (22)$$

A minimal phenomenological consistent extension is:

$$G^{(1)} \to G^{(1)} \exp\left[-\sigma^2 \int (\nabla \Sigma(x))^2 ds\right] \qquad (23)$$

Where σ parametetrizes sensitivity to variations in $\Sigma(x)$ along the optical path γ.

This approach is practical because of already existing ultra-high precision optical experiments like frequency comb spectroscopy [18], cavity QED [19] or polarization-sensitive interferometry [20]. They have reached levels where extremely minute decoherence or birefringence-like effects too can be constrained [21]. It can be examined if any residual, path-dependent coherence loss can be modelled via Σ-dependent corrections. Even null constraints would place bounds on σ and on admissible variations of $\Sigma(x)$.

Beyond this, in our present work we refrain from proposing experimental realizations for the $(3 + 1)$ regime. Although there have been several studies at the theoretical/hypothetical level on super-luminal entities involving tachyons and extended relativistic frameworks, no physical realizations have yet been made [22—23]. Furthermore, such studies suffer significant limitations like instabilities, vacuum decay etc. that inhibits experimental progress. That is why, we too keep it open for future investigations.

Endnotes:

1. Mathematically, this requires handling metrics that admit changing signature or rank reductions. We implement this by using a smooth scalar field $\sigma(x) \in [0,1]$ and projection operators that continually suppress components in direction that get inactive. See Section 3.4. for a covariant implementation.
2. Luminal null vectors exist in standard Lorentzian geometry as limit cases, assigning a different local dimension count to null trajectories is a novel modelling choice that we implement by the projection field P_t and by coupling the dimensional field ϕ to the null structure. This is consistent with the multi-time approach in which extra time-like d.o.f. is allowed for certain classes of excitations (see Sec. 3.7 for quantum wave-function formalism; also see [13]).
3. Metrics that change signature are a known scholarly subject (e.g. signature change cosmologies). The approach in this paper, however, implements a controlled signature change via the scalar ϕ and the projection operators. For well-posedness we will require the interpolation regions (where ϕ crosses thresholds) to be thin and handled with a regularization term in action.
4. The total action: $S = \int d^4x \sqrt{-g} \left(\frac{1}{16\pi G} R + \mathcal{L}_{\emptyset}\right)$ yield Einstein equations $G_{\mu\nu} = 8\pi G T_{\mu\nu}$. The stress-energy tensor includes contributions from scalar field and projection penalty form. Since, $\mathcal{L}_{proj}$ depends only algebraically (or at most fixed order) on the frame, it does not introduce higher-derivative instabilities.
5. The action is manifestly diffeomorphism invariant (assuming $\chi_A(\phi)$ and frame field choices are built covariantly e.g. via tetrad fields). Conservation of energy momentum follows via Bianchi identities and the ϕ field equations.

Appendix A.1. Geometric Definition of Temporal and Spatial Projections

Let $(\mathcal{M}, g)$ be a smooth $4 - D$ Manifold with metric $g_{\mu\nu}$ and let $\emptyset: \mathcal{M} \longrightarrow \mathbb{R}$ be a smooth scalar field determining the effective dimensionality pair $\left(D_t(\emptyset), D_s(\emptyset)\right)$ with $D_t + D_S = 4$. At each point $x \in \mathcal{M}$, define a decomposition of the tangent space:

$$T_x\mathcal{M} = T_x^{(t)}\mathcal{M} \oplus T_x^{(s)}\mathcal{M}$$

Where $dim\, T_x^{(t)} = D_t(\emptyset(x)$ and $dim\, T_x^{(s)} = D_s(\emptyset(x)$.

Let $\{u_I(x)\}_{I=1}^{D_t}$ be an orthonormal basis of the temporal subspace:

$$g(u_I, u_J) = -\delta_{IJ}$$

And $\{v_J(x)\}_{I=1}^{D_s}$ be an orthonormal basis of the temporal subspace:

$$g(v_J, v_K) = +\delta_{JK}$$

Then, we define the temporal projection operator:

$$P^{\mu}_{t\ \nu} = \sum_{I=1}^{D_t} u_I^{\mu} u_{I\nu}$$

And the spatial projector operator:

$$P^{\mu}_{s\ \nu} = \sum_{I=1}^{D_S} v_J^{\mu} v_{J\nu}$$

(ref. Eq. 19)

These satisfy:

1. Idempotence: $P_t^2 = P_t, P_s^2 = P_s$
2. Orthogonality: $P_t P_s = 0$
3. Completeness: $P_t + P_s = \delta_{\nu}^{\mu}$

Thus, the projection fields define a smooth decomposition of the tangent bundle consistent with the effective signature.

Appendix A.2. Variational origin of the Projectional Constraints

In the main text, we have enforced the projection structure through the term:

$$\mathcal{L}_{proj} = -\lambda \sum_A (\chi_A(\emptyset) g(e_A, e_A) - \eta_{AA})^2 \qquad (A.1)$$

We now provide a variational interpretation of this term.

At each point $x \in \mathcal{M}$**,** the scalar field $\emptyset(x)$ determines which directions are time-like and which are space-like, through the functions $\chi_A(\emptyset)$. Accordingly, we impose the condition:

$$\chi_A(\emptyset) g(e_A, e_A) = \eta_{AA} \qquad (A.2)$$

Where $\eta_{AA} \in \{-1, +1\}$ encodes the desired signature structure. This condition assigns causal character (time-like or space-like) to each basis direction in a manner controlled by $\emptyset(x)$.

Now, instead of enforcing (A.2) exactly as a hard constraint, we introduce a quadratic penalty functional:

$$\mathcal{E}[e_A, \emptyset] = \sum_A (\chi_A(\emptyset) g(e_A, e_A) - \eta_{AA})^2 \qquad (A.3)$$

The projection Lagrangian is then defined as:

$$\mathcal{L}_{proj} = -\lambda \mathcal{E}[e_A, \emptyset] \qquad (A.4)$$

With $\lambda > 0$ a coupling constant.

At the minimum of $\mathcal{L}_{proj}$, the condition (A.2) is satisfied, and the frame $\{e_A\}$ decomposes into time-like and space-like directions according to the value of $\emptyset(x)$.

This dynamically enforces the effective signatures $(D_t(\emptyset), D_s(\emptyset))$. In particular, when $\chi_A(\emptyset) \in \{0,1\}$, the constraint reduces to a sharp assignment of temporal and spatial directions, while smooth interpolations of $\chi_A(\emptyset)$ allow continues transitions between regimes.

Appendix A.3. Smooth Dependence on the Dimensionality Field $\emptyset(x)$

The functions $\chi_A(\emptyset)$ act as selection functions, assigning which basis directions are time-like. For mathematical consistency, we require

1. Piecewise-constant regimes: $D_t(\emptyset) \in \{1,2,3\}$
2. Smooth transitions (regularized): Instead of sharp step functions, use smooth approximations:

$$\chi_A(\emptyset) \approx \frac{1}{2}\left(1 + \tanh\left(\frac{\emptyset - \emptyset_A}{\epsilon}\right)\right)$$

With small $\epsilon > 0$.

This ensures smooth variation of $\mathcal{L}_{proj}$ and well-defined Euler-Lagrange equations. Thus, $P_t(x)$ and $P_s(x)$ become smooth tensor fields.

Appendix A.4. Scalar Field Dynamics and Stability

The scalar field evolves according to

$$\mathcal{L}_{\emptyset} = -\frac{1}{2}(\nabla\emptyset)^2 - V(\emptyset) + \mathcal{L}_{proj}(\emptyset, g, e_A) \qquad (A.5)$$

Now, to reproduce the three regimes, choose a multi-well potential:

$$V(\emptyset) = \alpha(\emptyset - \emptyset_1)^2(\emptyset - \emptyset_2)^2(\emptyset - \emptyset_3)^2$$

With minima corresponding to

$$\emptyset_1 \to (1,3), \emptyset_2 \to (2,2), \emptyset_3 \to (3,1)$$

Thus, $\emptyset(x)$ stabilizes at discrete values and each minimum corresponds to a fixed signature sector.

Varying $\emptyset$, we obtain:

$$\Box_g \emptyset - V'(\emptyset) + \frac{\partial \mathcal{L}_{proj}}{\partial \emptyset} = 0 \qquad (A.6)$$

The last term provides feedback coupling between geometry and dimensionality selection. Near a minimum $\emptyset = \emptyset_i, V''(\emptyset_i) > 0$. So, perturbations decay, ensuring that each regime is locally stable.

Appendix B.1: Existence of local adapted frame (Proposition 4.1)

Proposition (restated): Let $(\boldsymbol{\mathcal{M}}, g)$ be a smooth 4-dimensional manifold equipped with a smooth scalar field $\phi(x)$ such that effective dimensionality pair (D_t, D_s) is constant on an open, smoothly connected neighborhood $U \subset \boldsymbol{\mathcal{M}}$. Then there exists a smooth orthonormal frame field $\{e_A(x)\}_{A=0}^3$ on U adapted to the signature (D_t, D_s) .

Proof: Since g is a smooth, non-degenerate symmetric (0,2)-tensor field, each tangent space $T_x\boldsymbol{\mathcal{M}}$, $x \in U$, is a 4-dimensional real vector space equipped with a bilinear form of fixed index D_t.

By Sylvester's law of inertia, there exists, at each point x, a basis in which the metric takes the canonical diagonal form:

$$\eta_{AB} = \mathrm{diag}(\underbrace{-1, \dots, +1}_{D_t}, \underbrace{+1, \dots, -1}_{D_s})$$

Thus, at every point $x \in U$, there exists an orthonormal basis $\{e_A, e_B\} = \eta_{AB}$

Now since U is simply connected and g is smooth, the orthonormal frame bundle over U is a principal $O(D_t, D_s)$-bundle.

Standard results in differential geometry guarantee that such bundles admit smooth local sections (See [24]).

Alternatively, we can start with any smooth frame field $\{v_A(x)\}$ on U; then apply indefinite Gram-Schmidt orthonormalization pointwise (because the metric components $g_{\mu\nu}(x)$ and all algebraic operations (inner products, normalization, square roots of nonzero quantities) are smooth.

Thus, the projected orthonormal frame $\{e_A(x)\}$ depends smoothly on x.

Finally, the projection penalization term (Eq. 4)

$$\mathcal{L}_{proj} = -\lambda \sum_A (\chi_A(\emptyset) g(e_A, e_A) - \eta_{AA})^2$$

attains its minimum precisely when

$g(e_A, e_A) = \eta_{AA}$.

For all active directions A. Since $(\emptyset)$ is constant within the regime $\chi_A(\phi)$ is constant, ensuring consistency of the adapted frame hence (D_t, D_s) exists on U.

Appendix B.2. Proof of Proposition 4.2: Well posedness in Fixed Regime

Proposition (restated): In any region $U \subset M$ where (D_t, D_s) is constant, the coupled Einstein-ϕ system derived from the action forms a quasi-linear hyperbolic PDE system.

Proof: 1. Since (D_t, D_s) is constant on U, the metric $\theta_{\mu\nu}$ has fixed signature

$$(-,+,+,+), (-,-,+,+), \text{or } (-,-,-,+)$$

In each case, at least one time-like direction exists. Therefore, the principal symbol of the wave operator

$$\Box_g = g^{\mu\nu}\nabla_\mu\nabla_\nu$$

has hyperbolic character.

2. Einstein equations in harmonic gauge:
Choose harmonic coordinates satisfying:

$$\Box_g \, x^\mu = 0$$

In this gauge, Einstein equations reduce to

$$g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu} = F_{\mu\nu}(g, \partial g) + 8\pi G T_{\mu\nu}$$

which is a quasi-linear hyperbolic system.

3. The Scalar field satisfies:

$$\Box_g \, \phi - V'(\phi) + S_{\mathrm{proj}}(\phi, g) = 0$$

Here, the principal operator is $\Box_g$, hence hyperbolic, and lower-order terms do not affect hyperbolicity.

4. Projection term does not spoil hyperbolicity:

$$L_{\mathrm{proj}} = -\lambda \sum_A \, (x_A(\phi) g(e_A, e_A) - \eta_{AA})^2$$

This contributes to the algebraic terms in g, and at most first derivatives via frame dependence. Thus, it does not modify the principal symbol.

As per standard PDE theory, the coupled system $(g_{\mu\nu}, \phi)$ is second-order, quasi-linear, and hyperbolic. By standard theorems [16], such systems admit local existence, uniqueness, and continuous dependence on initial data.

Hence, the system is locally well-posed in any fixed-signature region.

References

[1] Weyl, H. (1918). Reine infinitesimalgeometrie. *Mathematische Zeitschrift*, *2*(3), 384-411.

[2] Wuensch, D. (2003). The fifth dimension: Theodor Kaluza's ground-breaking idea. *Annalen der physik*, *515*(9), 519-542.

[3] Klein, O. (1926). Quantentheorie und fünfdimensionale Relativitätstheorie. *Zeitschrift für Physik*, *37*(12), 895-906.

[4] Wuensch, D. (2009). Kaluza—Klein Theory. In *Compendium of quantum physics* (pp. 328-331). Berlin, Heidelberg: Springer Berlin Heidelberg.

[5] Dray, T., Ellis, G., Hellaby, C., & Manogue, C. A. (1997). Gravity and signature change. *General Relativity and Gravitation*, *29*(5), 591-597.

[6] Craig, W., & Weinstein, S. (2009). On determinism and well-posedness in multiple time dimensions. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, *465*(2110), 3023-3046.

[7] Myrzakulov, Y., Koussour, M., Karimov, M., & Rayimbaev, J. (2024). Signature flips in time-varying Λ (t) cosmological models with observational data. *The European Physical Journal C*, *84*(7), 665.

[8] Bars, I. (2001). Survey of two-time physics. *Classical and Quantum Gravity*, *18*(16), 3113.

[9] Vafa, C. (1996). Instantons on D-branes. *Nuclear Physics B*, *463*(2-3), 435-442.

[10] Bondarenko, S. (2022). Dynamical signature: complex manifolds, gauge fields and non-flat tangent space. *Universe*, *8*(10), 497.

[11] Gibbons, G. W., & Ishibashi, A. (2004). Topology and signature changes in braneworlds. *Classical and Quantum Gravity*, *21*(12), 2919.

[12] Christodoulou, M., Di Biagio, A., & Martin-Dussaud, P. (2022). An experiment to test the discreteness of time. *Quantum*, *6*, 826.

[13] Lienert, M., Petrat, S., & Tumulka, R. (2017, August). Multi-time wave functions. In *Journal of Physics: Conference Series* (Vol. 880, No. 1, p. 012006). IOP Publishing.

[14] Shestakova, T. P. (2021). On A. D. Sakharov's Hypothesis of Cosmological Transitions with Changes in the Signature of the Metric. *Universe*, *7*(5), 151.

[15] Calcagni, G. (2010). Fractal universe and quantum gravity. *Physical review letters*, *104*(25), 251301.

[16] Hughes, T. J., Kato, T., & Marsden, J. E. (1977). Well-posed quasi-linear second-order hyperbolic systems with applications to nonlinear elastodynamics and general relativity. *Archive for Rational Mechanics and Analysis*, *63*(3), 273-294.

[17] Abbott, B. P., Abbott, R., Abbott, T. D., Abernathy, M. R., Acernese, F., Ackley, K., ... & Cavalieri, R. (2016). Observation of gravitational waves from a binary black hole merger. *Physical review letters*, *116*(6), 061102.

[18] Picqué, N., & Hänsch, T. W. (2019). Frequency comb spectroscopy. *Nature photonics*, *13*(3), 146-157.

[19] Mivehvar, F., Piazza, F., Donner, T., & Ritsch, H. (2021). Cavity QED with quantum gases: New paradigms in many-body physics. *Advances in Physics*, *70*(1), 1-153.

[20] Wei, C. Q., Liu, J. B., Dong, Y. F., Sun, Y. N., Zhou, Y., Zheng, H. B., ... & Xu, Z. (2024). A polarization sensitive interferometer: Delta interferometer. *Chinese Physics B*, *33*(3), 034203.

[21] Wang, D. L., Wu, X. Z., Qin, R. Q., Han, J. T., Xie, P. P., Li, B. J., ... & Li, Y. F. (2026). Probing vacuum birefringence in an Ultrastrong Laser Field via High-energy Gamma-ray Polarimetry. *arXiv preprint arXiv:2603.05282*.

[22] Feinberg, G. (1967). Possibility of faster-than-light particles. *Physical Review*, *159*(5), 1089.

[23] Schwartz, C. (2022). A consistent theory of tachyons with interesting physics for neutrinos. *Symmetry*, *14*(6), 1172.

[24] Lee, J. M. (2003). Smooth manifolds. In *Introduction to smooth manifolds* (pp. 1-29). New York, NY: Springer New York.